In Defense of Trusts:  
R&D Cooperation in Global Perspective*  

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Abstract  
We re-examine the trade-off between the benefits of allowing firms to cooperate in R&D and the increased potential for product market collusion that R&D cooperatives bring about. For that we utilize a dynamic model of R&D whereby we do not restrict ourselves to a local analysis. That is, initial marginal costs may exceed the choke price such that R&D efforts can take place prior to production. Our framework yields four distinct scenarios, which includes the situation where firms continue to invest in developing further existing technologies although these technologies are destined to leave the market. We show that an extension of the cooperative agreement towards collusion in the product market is not necessarily welfare reducing: the full cartel develops further a wider range of initial technologies, it invests more in R&D such that process innovations are pursued more quickly, and it maintains a wider range of technologies.

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JEL: D43, D92, L13, L41, O31, O38

1 Introduction  
There are compelling reasons for rival firms to set up R&D cooperatives. These “organizations, jointly controlled by at least two participating entities, whose primary purpose is to engage in cooperative R&D” (Caloghirou et al., 2003) allow risks to be spread, secure better access to financial markets, pool resources such that economies of scale and scope in both research and development are better realized, and share costs. In the words of John Kenneth Galbraith (1952, pp. 86 – 87, emphasis added): “Most of the cheap and simple innovations have, to put it

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bluntly and unpersuasively, been made. Not only is development now sophisticated and costly but it must be on a sufficient scale so that success and failures will in some measure average out.” Moreover, R&D cooperatives internalize technological spillovers: the free flow of knowledge from the knowledge creator to its competitors.¹ Sustaining R&D cooperatives is thus perceived to diminish the failure of the market for R&D.²

However, as Scherer (1980) observes: “the most egregious price fixing schemes in American history were brought about by R&D cooperatives.”³ Martin (1995) shows that, indeed, the formation of R&D cooperatives makes it easier to sustain tacit collusion in the product market: “common assets create common interests, and common interests make it more likely that firms will non-cooperatively refrain from rivalrous behavior.” (Martin, 1995, p. 740).⁴ In this paper, we challenge the view that extending cooperative behavior to the product market necessarily diminishes total surplus.⁵

Static models of R&D predict total surplus to go down if members of an R&D cooperative collude in the product market.⁶ But a static view of the world necessarily ignores an important aspect of R&D: time. It takes time for an initial idea to be developed towards a marketable product; continuous process innovations gradually reduce production costs (Utterback, 1994). In this paper, therefore, we develop a dynamic model of R&D to examine the welfare implications of product market collusion by firms of an R&D cooperative.

Static models of R&D also predict that the marginal benefit of any R&D investment increases if firms collude in the product market. That is, firms are willing

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¹Bloom et al. (2007) estimate that a 10% increase in competitor’s R&D is associated with up to a 2.4% increase in firm’s own market value. Not surprisingly, internalizing technological spillovers is one of the prime reasons for firms to join an R&D cooperative (Hernan et al., 2003; see also Roeller et al., 2007).

²This motivates in particular why independent firms are allowed to cooperate in R&D. See Martin (1997) for an overview of the policy treatment of R&D cooperatives in the E.U., the U.S., and Japan.

³Goeree and Helland (2008) find that in the U.S. the probability that firms join an R&D cooperative has gone down due to a revision of antitrust leniency policy in 1993. This revision is perceived as making collusion less attractive. Goeree and Helland (2008) conclude that “Our results are consistent with RJVs [research joint ventures] serving, at least in part, a collusive function.”

⁴In a similar vein, Fisher (1990, p. 194) concludes that “…[firms] cooperating in R&D will tend to talk about other forms of cooperation. Furthermore, in learning how other firms react and just in living with each other, each cooperating firm will get better at coordination. Hence, competition in the product market is likely to be harmed.”

⁵Geroski (1992) argues that it is the feedback from product markets that directs research towards profitable tracks and that, therefore, for an innovation to be commercially successful there must be strong ties between marketing and development of new products. Jacquemin (1988) observes that R&D cooperatives are fragile and unstable. He reasons that when there is no cooperation in the product market, there exist a continuous fear that one partner in the R&D cooperative may be strengthened in such a way that it will become too strong a competitor in the product market. Preventing firms from collaborating in the product market may therefore destabilize R&D cooperatives, or prevent their creation in the first place.

⁶d’Aspremont and Jacquemin (1988) are the first to show that a scenario where firms cooperate in R&D and collude in the ensuing product market yields a lower total surplus than the situation where firms cooperate in R&D only.
to spend more resources on R&D if the intensity of product market competition is diminished through some collusive agreement.\textsuperscript{7} Put differently, any initial idea (that is, any initial level of marginal costs) is more likely to be developed further if firms collude in the product market. This suggests that no level of initial marginal costs of production should be excluded from the analysis, in particular marginal costs that exceed the choke price (that is, the maximum price at which the quantity sold is still positive).\textsuperscript{8} Moreover, requiring any time marginal costs to be below the choke price imposes that R&D activity and production coexist at all times. Surely this assumption is quite unlikely to hold for new technologies at their early stages of development. Indeed, research starts long before a prototype sees the light; development begins long before the launch of a new product. To properly assess the welfare implications of product market collusion induced by an R&D cooperative, this development phase should not be excluded from the analysis.

A distinguished feature of our approach, as compared to the received literature, is that we provide a \textit{global} analysis. That is, we consider all possible values of initial marginal costs, including those above the choke price (thus allowing research efforts to precede production), and we do not limit ourselves to equilibrium paths but consider all trajectories which satisfy necessary conditions (and that are as such candidates for an optimal solution). This enables us to determine the location of \textit{critical points} - points at which the optimal investment function qualitatively changes. That is, we determine the value of marginal costs for which R&D investments are terminated, and for which they are not initiated at all. As the position of these critical points differs between regimes, different regimes can lead to qualitatively different long-run solutions despite starting from the same initial technology.

The use of proper bifurcation theory\textsuperscript{9} allows us to model the R&D process prior to the production stage for all possible initial technologies. It yields a bifurcation diagram that indicates for every possible parameter combination the qualitative features of any market equilibrium. Our global analysis yields four distinct scenarios: (i) initial marginal costs are above the choke price and the R&D process is initiated; after some time production starts and marginal costs continue to fall

\begin{itemize}
\item\textsuperscript{7}Again, d’ Aspremont and Jacquemin (1988) are the first to show this formally. This touches upon the debate between Schumpeter Mark I (“...new combinations are, as a rule, embodied, as it were, in new firms which generally do not arise out of the old ones but start producing beside them...”; Schumpeter, 1934, p. 66) and Schumpeter Mark II (“As soon as we go into the details and inquire into the individual items in which progress was most conspicuous, the trail leads not to the doors of those firms that work under conditions of comparatively free competition but precisely to the doors of the large concerns...and a shocking suspicion dawns upon us that big business may have had more to do with creating that standard of living than with keeping it down”; Schumpeter, 1943, p. 82).
\item\textsuperscript{8}Here we deviate from the related literature that, with no exception, restricts the analysis to initial levels of marginal costs that are below the choke price (cf. Petit and Tolwinski (1999), Cellini and Lambertini (2009), Lambertini and Mantovani (2009) and Kovac \textit{et al.}(2010)).
\item\textsuperscript{9}The variations in the values of parameters typically lead to qualitative changes in the solution structure (e.g., some steady states lose their stability, indifference points appear). Such qualitative changes in the solution structure due to smooth variations in parameters are called bifurcations. For an introduction, see Grass \textit{et al.} (2008), or Kiseleva and Wagener (2011).
\end{itemize}
with subsequent R&D investments; (ii) initial marginal costs are above the choke price and the R&D process is not initiated, yielding no production at all; (iii) initial marginal costs are below the choke price and the R&D process is initiated; production starts immediately and marginal costs continue to fall over time, and (iv) initial marginal costs are below the choke price and the initiated R&D process is progressively scaled down; production starts immediately but the technology (and production) will die out over time; the firms leave the market.

We then compare two different regimes across these scenarios. In the first regime, firms cooperate in R&D while remaining competitors on the concomitant product market. In the second regime, the R&D cooperative is extended to the product market; firms form a full cartel (this case closely follows Hinloopen, Smrkolj and Wagener (2011), where we develop the global framework for a monopolist). We then compare the qualitative properties of these two regimes in order to assess the potential set-back of R&D cooperatives in that they can serve as a platform to coordinate prices.

In sum, we find that: (i) the range of initial marginal costs which (eventually) yields to production is larger when firms of the R&D cooperative also collude in the product market, (ii) product market collusion accelerates the speed with which new technologies enter the product market, and (iii) the set of initial marginal costs that induce firms to abandon the technology in time is larger if firms cannot collude in the product market.

Our results are not without policy implications. When designing antitrust policies it is important to understand that these policies not only affect current markets, but also markets that are not yet visible. In particular, preventing firms from colluding in the product market reduces the number of potential R&D trajectories that successfully lead to new markets. In itself this constitutes a welfare loss.

The remainder of the paper is organized as follows. Section 2 presents the model. Section 3 describes all possible equilibria in case firms cooperate in R&D, and analyzes the properties of the global equilibrium dynamics. Section 4 derives all equilibria in case firms collude both in R&D and on the product market, which are confronted with the full collusion case in Section 5. Section 6 concludes.

2 The model

Time $t$ is continuous: $t \in [0,\infty)$. There are two a priori fully symmetric firms which both produce a homogenous good at constant marginal costs. In every instant, market demand is:

$$p(t) = A - q_i(t) - q_j(t),$$

where $p(t)$ is the market price, $q_i(t)$ is the quantity produced by firm $i = \{1, 2\}$, and $A$ is the choke price.

Each firm can reduce its marginal cost by investing in R&D. In particular, firm $i$ exerts R&D effort $k_i(t)$ and as a consequence of these investments, its marginal
cost evolves over time as follows:

\[
\frac{dc_i(t)}{dt} = \dot{c}_i(t) = c_i(t) (-k_i(t) - \beta k_j(t) + \delta),
\]

where \( k_j(t) \) is the R&D effort exerted by its rival and where \( \beta \in [0, 1] \) measures the degree of spillover. The parameter \( \delta > 0 \) is the constant rate of decrease in efficiency due to the ageing of technology. Both firms have an identical initial technology \( c_i(0) = c_j(0) = c(0) \), which is drawn by Nature. The cost of R&D efforts per unit of time is for firm \( i \) given by:

\[
\Gamma_i(k_i(t)) = b(k_i(t))^2,
\]

where \( b > 0 \) is inversely related to the cost-efficiency of the R&D process. Hence, the R&D process exhibits decreasing returns to scale.\(^\text{10}\) Both firms discount the future with the same constant rate \( \rho > 0 \). Instantaneous profit of firm \( i \) is:

\[
\pi_i(q_i, q_j, k_i, c_i) = (A - q_i - q_j - c_i)q_i - bk_i^2,
\]

yielding total discounted profit:

\[
\Pi_i(q_i, q_j, k_i, c_i) = \int_0^\infty \pi_i(q_i, q_j, k_i, c_i)e^{-\rho t} dt.
\]

The model has five parameters: \( A, \beta, b, \delta, \) and \( \rho \). The analysis can be simplified by considering a rescaled version of the model which, as defined in Lemma 1, carries only three parameters: \( \beta, \phi, \) and \( \tilde{\rho} \).

**Lemma 1.** By choosing the units of \( t, q_i, q_j, c_i, c_j, k_i, \) and \( k_j \) appropriately, we can assume \( A = 1, b = 1, \) and \( \delta = 1 \). This yields the following rescaled version of the model:

\[
\tilde{\pi}_i(\tilde{q}_i, \tilde{q}_j, \tilde{k}_i, \tilde{c}_i) = (1 - \tilde{q}_i - \tilde{q}_j - \tilde{c}_i)\tilde{q}_i - \tilde{k}_i^2,
\]

\[
\tilde{\Pi}_i(\tilde{q}_i, \tilde{q}_j, \tilde{k}_i, \tilde{c}_i) = \int_0^\infty \tilde{\pi}_i(\tilde{q}_i, \tilde{q}_j, \tilde{k}_i, \tilde{c}_i)e^{-\tilde{\rho} t} d\tilde{t}
\]

\[
\dot{\tilde{c}}_i = \tilde{c}_i \left( 1 - (\tilde{k}_i + \beta \tilde{k}_j) \phi \right), \quad \tilde{c}_i(0) = \tilde{c}_0, \quad \tilde{c}_i \in [0, \infty) \forall \tilde{t} \in [0, \infty)
\]

\[
\tilde{\rho} > 0, \quad \phi > 0
\]

with conversion rules: \( \tilde{q}_i = A\tilde{q}_i, \tilde{q}_j = A\tilde{q}_j, \tilde{k}_i = A\tilde{k}_i, \tilde{k}_j = A\tilde{k}_j, \tilde{c}_i = A\tilde{c}_i, \tilde{c}_j = A\tilde{c}_j, \pi_i = A^2\tilde{\pi}_i, \pi_j = A^2\tilde{\pi}_j, \phi = \frac{A}{\delta v_b}, \tilde{\rho} = \frac{\rho}{\delta}.

The rescaled version of the model introduces a new parameter \( \phi \), which captures the efficiency of the R&D process: a higher \( b \) implies that each unit of R&D effort costs more, whereas a higher \( \delta \) implies that each unit of R&D effort decreases the marginal cost by less. Therefore, a higher (lower) \( \phi \) corresponds to a more (less) efficient R&D process. For notational convenience we henceforth omit tildes.

\(^{10}\)In assuming decreasing returns to scale and a positive \( \delta \), we follow the bulk of the literature. See Hinloopen, Smrkolj and Wagener (2011) for a discussion of these assumptions.
3 The R&D Cartel

In this section, we assume that each firm operates its own R&D laboratory and production facility. However, while firms select their output levels non-cooperatively, they adopt a strictly cooperative behavior in determining their R&D efforts so as to maximize their joint profits. These assumptions amount to imposing a priori the symmetry condition $k_i(t) = k_j(t) = k(t)$.$^{11}$ As $c_i(0) = c_j(0) = c(0)$, this implies that $c_i(t) = c_j(t) = c(t)$. Equation (8) thus reads as$^{12}$:

$$\dot{c} = c(1 - (1 + \beta)\phi k).$$

(11)

The instantaneous profit of firm $i$ is:

$$\pi_i(q_i, q_j, k, c) = (1 - q_i - q_j - c)q_i - k^2,$$

yielding its total discounted profit over time:

$$\Pi_i(q_i, q_j, k, c) = \int_0^\infty \pi_i(q_i, q_j, k, c)e^{-\rho t}dt.$$  

(13)

We look for feedback (also called closed-loop) Nash equilibria. As firms cooperatively decide on their R&D efforts, the only competitive decisions are that of production levels. However, as quantity variables do not appear in the equation for the state variable (11), production feedback strategies of a dynamic game are simply static Cournot-Nash strategies of each corresponding one-period game.

Solving $\max_{q_i \in \mathbb{R}^+} \pi_i$ gives us standard Cournot best-response functions for the product market:

$$q_i(q_j) = \begin{cases} 
\frac{1}{2}(1 - c - q_j) & \text{if } q_j < 1 - c, \\
0 & \text{if } q_j \geq 1 - c,
\end{cases}$$

(14)

where the second part comes from the boundary condition for the non-negative production levels $q_{i,j} \geq 0$. Solving for the Cournot-Nash production levels, we obtain

$$q^N = \begin{cases} 
\frac{1}{3}(1 - c) & \text{if } c < 1, \\
0 & \text{if } c \geq 1.
\end{cases}$$

(15)

$^{11}$In this paper we are looking only for symmetric equilibria. However, it is perhaps possible that joint profits could be increased if firms made unequal investments and produce unequal amounts. Such an asymmetric strategy would raise investment costs (the R&D cost function is convex) but might create more than offsetting benefits in the production stage. Salant and Shaffer (1998) consider this possibility within the framework of a static model of strategic R&D.

$^{12}$It may seem reasonable to assume that when firms cooperate in R&D they also fully share information, that is, that the level of spillover is at its maximum ($\beta = 1$); see Kamien et al., 1992. For the sake of generality, we do not a priori fix the value of $\beta$ at its maximum. There are also intuitive arguments for not doing so as there might still be some ex post duplication and/or substitutability in R&D outputs if firms operate separate laboratories.
Consequently, the instantaneous profit of each firm is

$$\pi(c,k) = \begin{cases} \frac{1}{9}(1-c)^2 - k^2 & \text{if } c < 1, \\ -k^2 & \text{if } c \geq 1. \end{cases} \quad (16)$$

Then, the optimal control problem of the R&D cartel is to find control $k^*$ (R&D effort of each firm) that maximizes the total discounted joint profit of the two firms taking into account the state equation (11), initial condition $c(0) = 0$, and the boundary condition $k \geq 0$ which must hold at all times. Note that according to (11), if $c_0 > 0$, then $c(t) > 0$ for all $t$. The set of all possible states at each time $t$ is given by $c \in [0, \infty)$.\(^{13}\)

To solve the dynamic optimization problem, we introduce the current-value Pontryagin function (also called pre-Hamilton or un-maximized Hamilton function)\(^{14}\)

$$P(c,k,\lambda) = \begin{cases} \frac{1}{9}(1-c)^2 - k^2 + \lambda c(1 - (1 + \beta)\phi k) & \text{if } c < 1, \\ -k^2 + \lambda c(1 - (1 + \beta)\phi k) & \text{if } c \geq 1, \end{cases} \quad (17)$$

where $\lambda$ is the current-value co-state variable of a firm in the R&D cartel. It measures the marginal worth of the increment in the state $c$ for each firm in the cartel at time $t$ when moving along the optimal path. As marginal cost is a "bad", we expect $\lambda(t) \leq 0$ along optimal trajectories.

We use Pontryagin’s Maximum Principle to obtain the solution to our optimization problem (see Appendix B for derivation). This solution can be expressed in the form of a state-control system which has two regimes (following the two part composition of the Pontryagin’s function). The first one corresponds to $c < 1$ and is characterized by positive production ($q = (1-c)/3$). The second one corresponds to $c \geq 1$ and is characterized by zero production.\(^{15}\) The state-control system with positive production consists of the following two differential equations\(^{16}\):

\(^{13}\)In other words, firms that jointly decide on their R&D efforts know that as the result of this cooperation, the marginal cost of both firms will be the same and that their product market profits directly depend on it (recall (16)). Knowing this, they will select such a time path of R&D efforts, and through it the path of their marginal cost over time, that their joint profits, which in our symmetric case they share equally, will be the highest possible.

\(^{14}\)We omit a factor of 2 for combined profits to obtain the solution expressed in per-firm values. Due to symmetry, maximizing the per-firm total profit is the same as maximizing the two firms’ combined total profit.

\(^{15}\)Recall from Lemma 1 that $A = 1$ in the rescaled model. In the original (non-rescaled) model, the analogous conditions for positive and zero production are $c(t) < A$ and $c(t) \geq A$, respectively.

\(^{16}\)Our closed-loop solution differs from that of Cellini and Lambertini (2009), who consider the case when marginal cost is always lower than the choke price. This is so because their proof that the open-loop and closed-loop solutions coincide is flawed by the fact that in their derivation of the closed-loop solution, players’ output choices are not properly treated as functions of the state variable. The derivations of the authors implicitly assume that if marginal cost within the R&D cartel changes, the opponent’s quantity does not change, which is the violation of the feedback principle underlying the closed-loop solution. It is also counterintuitive as firms in the R&D cartel

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\[
\begin{aligned}
\dot{k} &= \rho k - \frac{(1+\beta)\varphi}{9} c(1-c), \\
\dot{c} &= c(1 - (1 + \beta)\varphi k).
\end{aligned}
\] (18)

The state-control system with zero production is given by
\[
\begin{aligned}
\dot{k} &= \rho k, \\
\dot{c} &= c(1 - (1 + \beta)\varphi k).
\end{aligned}
\] (19)

3.1 Steady-state solutions

The steady-state solutions to (18) and (19) are obtained by imposing the stationarity conditions \(\dot{k} = 0\) and \(\dot{c} = 0\).

Lemma 2. The “no-production” state-control system (19) has no steady state in the region where it is defined.

Proof. See Appendix C.

Consider now system (18). Imposing the stationarity condition \(\dot{k} = 0\), we obtain
\[
k^{RC} = \frac{(1 + \beta)\varphi}{9\rho} c(1 - c) \geq 0 \quad \forall c \in [0, 1),
\] (20)

where superscript \(RC\) stands for R&D Cartel.\(^{17}\) Steady-state marginal cost follows from inserting (20) into the state dynamics (11) and imposing the stationarity condition \(\dot{c} = 0\):
\[
\dot{c} = c \left(1 - \frac{(1 + \beta)^2\varphi^2}{9\rho} c(1-c)\right) = 0.
\] (21)

This yields:
\[
c^{RC} = 0, \quad c^{RC} = \frac{1}{2} \pm V,
\] (22)

where
\[
V = \frac{1}{2} \sqrt{1 - \frac{36\rho}{(1 + \beta)^2\varphi^2}}.
\] (23)

Observe that \(V\) is real if and only if \(\phi \geq 6\sqrt{\rho}/(1 + \beta)\), in which case \(V \in [0, \frac{1}{2})\) and \(c^{RC} \in [0, 1)\). The solution to the system is summarized in Proposition 1.

\(^{17}\)All steady-state values have superscript \(RC\).

are supposed to jointly decide on their R&D effort taking into account that marginal cost in any period affects the Nash-equilibrium profit in the product market. This profit is a function of the cartel’s marginal cost and reflects changes in outputs of both players. Our calculations also show that their solution leads to situations in which the profit of the R&D cartel can be higher than that of the full cartel. We shall question any such a result as the full cartel can in principle always do what a lower form of cooperation (R&D cartel) does if this brings about a higher profit. We therefore stick to our closed-loop solution, which does not suffer from this irregularity, for all future comparisons between regimes.
Proposition 1. If \( \phi > 6\sqrt{\rho}/(1 + \beta) \), the state-control system with positive production (30) has three steady states:

i) \((c_{RC}, k_{RC}) = (0, 0)\) is an unstable node,

ii) \((c_{RC}, k_{RC}) = \left( \frac{1}{2} + V, \frac{1}{(1+\beta)\phi} \right)\) is either an unstable node or an unstable focus, and

iii) \((c_{RC}, k_{RC}) = \left( \frac{1}{2} - V, \frac{1}{(1+\beta)\phi} \right)\) is a saddle-point steady state.

At \( \phi = 6\sqrt{\rho}/(1 + \beta) \), a so-called “saddle-node bifurcation” occurs as the last two steady states collide and form a so-called “semi-stable” steady state.

If \( \phi < 6\sqrt{\rho}/(1 + \beta) \), the state-control system with positive production has one single steady state: the origin \((c_{RC}, k_{RC}) = (0, 0)\), which is unstable.

3.2 Global Optimality

The stable manifold of the saddle-point steady state is one of the candidates for an optimal solution. As the following remark shows, it may not (always) be the true optimum.\(^{18}\)

Remark 1. The Pontryagin function of the optimization problem as given in (29) is not jointly concave in the state and control variables. Hence, necessary conditions for an optimum are not necessarily sufficient.

Proof. See Appendix E. \(\square\)

To analyze global optimality, we have to inspect the complete state-control space, which is the set of solutions \((c(t), k(t))\) of the state-control system considered as parameterized curves (trajectories) in a plane. Its general form is sketched in Figure 1.

The dotted vertical line \( c = 1 \) separates the region with zero production from the region with a positive production. In the left-side region, the parabola which achieves its maximum at \( c = 1/2 \) is the locus \( \dot{k} = 0 \). The loci representing points where \( \dot{c} = 0 \) are the horizontal line with height \( \frac{1}{(1+\beta)\rho} \) and the k-axis. Loci \( \dot{c} = 0 \) and \( \dot{k} = 0 \) intersect in the saddle-point steady state \((S_2)\) and the two unstable steady states \((S_1)\) and \((S_3)\). The black arrows in squares indicate the direction of trajectories in each respective region. The horizontal arrows indicate an increase or a decrease in \( c \), whereas the vertical ones refer to changes in \( k \). A number of trajectories is indicated by grey arrows. Jointly, the trajectories cover the entire

\(^{18}\)It is worthwhile emphasizing that concavity of the objective function with respect to the control variable is not sufficient to guarantee that the stable manifold is, in fact, an optimal solution. As stated in Remark 1, sufficiency requires concavity of the Pontryagin function in the state variable as well. See Grass et al. (2008) for further details.
parameter space. However, there are only two trajectories leading to the saddle-point steady state; these are called “stable paths” or “stable manifolds”. They are indicated by the connected black arrows pointing towards $S_2$. The lighter arrows pointing away from $S_2$ indicate the two unstable paths. Trajectories spiral out from $S_3$ in a counter-clockwise motion.

The solution to the state-control system (31), which gives trajectories in the region with zero production, is given by:

\[
k(t) = C_1 e^{\rho t},
\]

\[
c(t) = C_2 e^{\left(t - \frac{e^{\rho (1+\beta) \phi} C_1}{\rho}\right)},
\]

where $C_1$ and $C_2$ are positive real constants.

A solution to the state-control system (30), which yields trajectories in the region with positive production, cannot be obtained analytically; we therefore make use of geometrical-numerical techniques.  

A particular trajectory is a candidate for an optimal solution if it satisfies all necessary conditions: those given by Pontryagin’s Maximum Principle together with transversality conditions (see Appendix B). As Figure 1 shows, there exists a whole range of solutions $(c(t), k(t))$ to the state-control systems. These trajectories all satisfy the conditions of Pontryagin’s Maximum Principle. By ruling out those

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19 All numerical calculations were done in Matlab.
trajectories that do not satisfy the transversality conditions, we are left with the candidates for an optimal solution.

**Lemma 3.** The points in the set

\[ \{(c, k) : c \in (0, 1), k = 0\} \]

cannot be a part of any optimal trajectory.

*Proof.* See Appendix F.

**Lemma 4.** All trajectories along which \( k \to \infty \) and \( c \to 0 \) as \( t \to \infty \) can be ruled out as optimal solutions.

*Proof.* See Appendix G.

For instance, the trajectory denoted by \( L_1 \) in Figure 1 is in the set of Lemma 4.

**Lemma 5.** The trajectory through the point \((c, k) = (1, 0)\) satisfies the transversality conditions (40) and (41).

*Proof.* See Appendix H.

In Figure 1, the trajectory of Lemma 5 is labeled \( L_3 \). We call this the “exit trajectory” as it implies that the firms (eventually) exit the market. Indeed, at some stage the exit trajectory enters the region where \( c \geq 1 \), at which point both R&D investment and production come to a halt.\(^{20}\)

Notice that the stable path (denoted by \( L_2 \) in Figure 1) to the saddle-point steady state \((S_2)\) also satisfies the transversality conditions. Notably, along this trajectory, both \( c \) and \( k \) converge to finite limits, and then so does the costate variable as well.

As the following corollary to the above lemmas shows, we are left with two candidates for an optimal solution.

**Corollary 1.** The set of candidates for an optimal solution consist of the stable path of the saddle-point steady state and the trajectory through the point \( \{(c, k) = (1, 0)\} \).

*Proof.* See Appendix I.

To assess the dependence of the solution structure on the model parameters, we carry out a detailed bifurcation analysis. This analysis results in a bifurcation diagram, provided in Figure 2, which shows the optimal R&D investments for varying parameter values.\(^{21}\) Every combination of the exogenous parameters yields a point

\(^{20}\)Strictly speaking, firms’ marginal costs continue to increase along the exit trajectory due to the positive depreciation rate \( \delta \). However, we interpret any situation in which a firm stays inactive as if it has left the market.

\(^{21}\)The bifurcation diagram in Figure 2 is drawn for the parameters of the non-rescaled model.
on this diagram. It consists of five distinct regions, each representing a particular structure of the set of optimal solutions. In what follows, we limit ourselves to briefly explaining some of its more important parts and refer the reader to Hinloopen, Smrkolj and Wagener (2011) for a detailed exposition of it.

Figure 2: Bifurcation diagram for $\beta = 1$. The uppermost curve represents parameter values for which the indifference point is exactly at $c = 1$. SN stands for the “saddle-node bifurcation curve”, SN’ for the “inessential saddle-node bifurcation curve”, IA for the “indifference-attractor bifurcation curve”, and IR for the “indifference-repeller bifurcation curve”. ISN indicates the “indifference-saddle-node point”, which is the point from which indifference-attractor, indifference-repeller, saddle-node and inessential saddle-node curves emanate. It’s coordinates are $\tilde{\rho} \approx 2.14$ and $\phi \approx 8.777$. Roman numerals (I-V) indicate the corresponding parameter regions discussed in the main text. The shaded region indicates the parameter region for which firms always exit the market. Notice that the axes are excluded from the admissible parameter space. The curve representing indifference points at $c = 1$ obtains a value of $\phi \approx 2.998$ for $\hat{\rho} = 1 \times 10^{-5}$.

In general, there are two possibilities. Either the exit trajectory is always optimal (shaded region) or it is optimal only for some subset of the state space (non-shaded region). In the first case, the (new) technology is always phased out by firms (eventually) exiting the market. In the second case, there exist a threshold point: the stable path of the saddle-point steady state is optimal if the initial marginal cost is below some threshold level, whereas the exit trajectory is optimal if initial marginal cost is above that level. This implies, in particular, that convergence to
the saddle-point steady state along the stable path can be locally optimal only; it
is never optimal globally. The threshold point \( \hat{c} \) can be either an indifference
point or a repeller. Indifference points are initial states \( c = c_0 \) for which the firm
is indifferent between the two possible investment policies (i.e., the stable path
and the exit trajectory). A repeller differs from an indifference point in that at a
repelling point, it is optimal to stay at that point forever after.

In Region I, there is an indifference point above one. Therefore, the system
will for sure converge to the saddle-point steady state for all initial technologies
\( c_0 \in (0, 1] \). A typical example is depicted in Figure 3. In plot (a), the arrows
indicate the direction of motion along optimal trajectories. The two regions (pro-
duction, no production) are split by the dotted vertical line \( c = 1 \). For a given \( \rho \),
Region I is characterized by a relatively high \( \phi \) (and \( \beta \)), that is, by relatively large
demand and/or high R&D efficiency. In such a favorable environment, the firms
can compensate for their early stage losses if, initially, production is postponed as
that would yield negative mark-ups. There exists, however, a finite upper bound
\( \hat{c} > 1 \) (indicated by a dashed vertical line) for the initial marginal costs \( c_0 \) beyond
which future profits are not enough to compensate for short-run losses \( (\hat{c} \approx 1.3
in the case considered). In that case, the technology will never enter the market.

As indicated in plot (a), for all initial marginal costs up to \( c_0 \approx 1.3 \), the optimal
solution is the stable path leading to the saddle-point steady state. In Plots (b)–(d)
we see that when the firms start with initial marginal cost above 1, the optimal
decision leads the firms to first produce nothing but to invest increasingly in the
reduction of their marginal costs. When marginal costs are below one, they start
producing. The level of R&D investments then gradually decreases to its long-
run steady-state level. Plot (e) shows the total discounted profit corresponding
to different initial marginal costs when following the optimal path. The point of
indifference occurs where the total profit function obtains a value of zero (which
is the value of the total profit function for the exit trajectory in the region \( c \geq 1 \).
As the total profit beyond this point is negative (the firms make a net loss), the
firms prefer no investment at all (giving them total profits of zero). It’s worth to
emphasize that having a phase of zero production as a part of the stable path is
possible only if the range of initial marginal costs is not bounded to be below the
choke price. In this phase instantaneous profits are negative, but total discounted
profits are positive; the firms invest in R&D because once marginal costs are below
the choke price, production becomes profitable. As shown in plot (e), the value of
the total discounted profit function is increasing with decreasing marginal cost.
Clearly, starting with lower initial marginal costs allows each firm in the cartel to
obtain a higher present discounted value of its profit flow. In sum, in Region I

\(^{22}\)The saddle-point steady state does not even exist for all parameter values (recall Proposition 2).
\(^{23}\)Also called Skiba points or DNSS points.
\(^{24}\)Note that \( c_0 = 0 \) corresponds to an unstable steady state; for this trivial value of the marginal
cost, the firms stay at that point forever.
\(^{25}\)Plots (b)–(d) have as their starting value the indifference point. Their shape is, though, the same
for every \( c_0 \in (1, \hat{c}] \).

13
Figure 3: State-control space (a), time paths for quantity (b), marginal cost (c), and R&D effort (d), and the correlation diagram between the total discounted profit and marginal cost (e), respectively. All plots show curves for parameter values \((\beta, \rho, \phi) = (1, 0.5, 4)\). Variables are in per-firm values.

we always have an indifference point at some \(c = \hat{c} > 1\) and, in particular, the convergence to the saddle-point steady-state is for these parameters guaranteed for \(c_0 \in (0, 1]\), but not for all \(c_0 \in (1, \infty)\). The exact position of the indifference point is defined in Lemma 6.

Lemma 6. If an indifference point between the stable path and the exit trajectory does exist in the region with zero production \((c \geq 1)\), it is given by the value of the marginal cost \(\hat{c} \geq 1\) for which the point \((c, k) = \left(\hat{c}, \frac{2}{(1+\beta)^\phi}\right)\) lies on the stable path. For \(c_0 = \hat{c}\), total discounted profits are zero for both trajectories.

Proof. See Appendix J. \(\square\)

Region I is bounded from below by the curve representing those parameter combinations for which the indifference point is exactly at \(c = 1\). Passing through this curve, we enter Region II. There, the indifference point \(\hat{c}\) is below \(c = 1\). This implies in particular that at the indifference point the firms produce a positive

\[related\text{ }equation\]

Region III is similar to Region II in that in both cases the threshold point is below \(c = 1\). The small qualitative difference between the two regions is that in Region III, the threshold point is not an indifference point but a repeller. Region III is separated from Region II by a type-2 indifference-repeller bifurcation curve \(IR_1(2)\), at which a repeller changes into an indifference point (see Hinloopen, Smrkolj, and Wagener (2011)).
quantity. The left plot in Figure 4 illustrates this case. For all initial technology levels corresponding to marginal costs below \( \hat{c} \) (for this particular parameters, we have \( \hat{c} \approx 0.52 \)), the firms follows the stable path. Although it is also profitable to invest in R&D for some initial technology level \( c_0 \in (\hat{c}, 1) \), the technology is not promising enough for the firms to select the stable path towards the saddle-point steady-state. Rather, they invests in R&D at some smaller rate that retards the decay of the technology level optimally; but eventually the technology will leave the market, and the firms with it. For an even higher initial level of marginal cost \( (c_0 \geq 1) \), the firms do not initiate any activity; the new technology is not developed at all and the firms do not enter the market.

![Figure 4: State-control space for parameters in Region II (left plot) and in the shaded region (right plot). The left plot is drawn for parameter values \((\beta, \rho, \phi) = (1, 1, 3.5)\), the right plot for\((\beta, \rho, \phi) = (1, 1, 2.5)\).](image)

Region IV and Region V, which together form the shaded region, both represent the parameter region for which there is never convergence to the saddle-point steady state because it is either not optimal to approach it (Region V)\(^{27}\) or because there is no saddle-point steady state at all (Region IV). This case is illustrated in the second plot of Figure 4. The optimal solution is the exit trajectory: for \( c_0 \geq 1 \) the firms never initiate any activity, while for \( c_0 \in (0, 1) \) the firms do invest in R&D but they will leave the market at some future instant. This is intuitive as in the shaded region we have that \( \phi \) is relatively small (for a given \( \rho \) and \( \beta \)). That is, R&D is relatively inefficient and/or market demand is relatively small. Note that in the second plot of Figure 4, R&D investments are initially increasing over time for very low values of marginal costs. These investments reduce the speed at which the firms will leave the market; for low initial marginal cost it turns out to be profitable to slow down this process in order to profit optimally from low marginal costs.

Moving from the shaded region, where the firms always exit the market, by increasing \( \phi \) while keeping \( \rho \) and \( \beta \) fixed, we arrive at Region II, where staying in the market is possibly an optimal solution depending on the initial level \( c_0 \) of the marginal cost. The interpretation is straightforward: staying in the market becomes

\(^{27}\)While the stable-path does exist in this region, it brings about a lower total discounted profit than the exit trajectory.
an option if R&D becomes more efficient and/or if market demand increases.

4 R&D and market cooperation (Full Cartel)

In this section, we again assume that each firm operates its own R&D laboratory and production facility. This time, however, firms adopt a strictly cooperative behavior in determining both their R&D efforts and output levels. These assumptions amount to imposing \textit{a priori} the symmetry conditions \( k_i(t) = k_j(t) = k(t) \) and \( q_i(t) = q_j(t) = q(t) \). Equation (8) thus reads as:

\[
\dot{c} = c(1 - (1 + \beta)\phi k). \tag{26}
\]

The profit of each firm in every instant is:

\[
\pi(q, k, c) = (1 - 2q - c)q - k^2, \tag{27}
\]

yielding its total discounted profit over time:

\[
\Pi(q, k, c) = \int_0^\infty \pi(q, k, c)e^{-\rho t} dt. \tag{28}
\]

The optimal control problem of the two firms is to find controls \( q^* \) and \( k^* \) that maximize the profit functional \( \Pi \) subject to the state equation (26), the initial condition \( c(0) = c_0 \), and two boundary conditions which must hold at all times: \( q \geq 0 \) and \( k \geq 0 \). Note that according to (26), if \( c_0 > 0 \), then \( c(t) > 0 \) for all \( t \). The set of all possible states at each time \( t \) is given by \( c \in [0, \infty) \).

To solve the dynamic optimization problem, we introduce the current-value Pontryagin function:

\[
P(c, q, k, \lambda) = (1 - 2q - c)q - k^2 + \lambda c(1 - (1 + \beta)\phi k), \tag{29}
\]

where \( \lambda \) is the current-value co-state variable. It measures the marginal worth of the increment in the state \( c \) for a firm in the cartel at time \( t \) when moving along the optimal path. As marginal cost is a “bad”, we expect \( \lambda(t) \leq 0 \) along optimal trajectories.

To obtain the solution to our optimization problem, we follow the same procedure as in the case of the R&D cartel (see Appendix B for derivation). This solution can again be expressed in the form of a state-control system which has two regimes. The first one corresponds to \( (c < 1) \) and is characterized by positive production \( (q = (1 - c)/4) \). The second one corresponds to \( (c \geq 1) \) and is characterized by zero production.

\[28\text{Again, due to symmetry, maximizing per-firm total profit is the same as maximizing the two firms’ combined total profit.}\]
The state-control system with positive production consists of the following two differential equations:

\[
\begin{align*}
\dot{k} &= \rho k - \frac{(1+\beta)\phi}{8}c(1-c) \\
\dot{c} &= c(1 - (1 + \beta)\phi k)
\end{align*}
\]  

(30)

The state-control system with zero production is given by

\[
\begin{align*}
\dot{k} &= \rho k \\
\dot{c} &= c(1 - (1 + \beta)\phi k)
\end{align*}
\]

(31)

4.1 Steady-state solutions

The steady-state solutions to (30) and (31) are obtained by imposing the stationarity conditions \(\dot{k} = 0\) and \(\dot{c} = 0\).

**Lemma 7.** The “no-production” state-control system (31) has no steady-state in the region where it is defined.

**Proof.** See Appendix C.

Consider now system (30). Imposing the stationarity condition \(\dot{k} = 0\), we obtain

\[
k^{FC} = \frac{(1 + \beta)\phi}{8\rho}c(1 - c) \geq 0 \quad \forall c \in [0, 1],
\]

(32)

where superscript FC stands for Full Cartel. Steady-state marginal cost follows from inserting (32) into the state dynamics (26) and imposing the stationarity condition \(\dot{c} = 0\). This yields:

\[
c^{FC} = 0, \quad c^{FC} = \frac{1}{2} \pm V,
\]

(33)

where

\[
V = \frac{1}{2} \sqrt{1 - \frac{32\rho}{(1 + \beta)^2\phi^2}}.
\]

(34)

Observe that \(V\) is real if, and only if, \(\phi \geq \sqrt{32\rho}/(1 + \beta)\), in which case \(V \in [0, \frac{1}{2}]\) and \(c^{FC} \in [0, 1]\). The solution to the system is summarized in Proposition 2.

**Proposition 2.** If \(\phi > \sqrt{32\rho}/(1 + \beta)\), the state-control system with positive production (30) has three steady states:

i) \((c^{FC}, k^{FC}) = (0, 0)\) is an unstable node,

ii) \((c^{FC}, k^{FC}) = \left(\frac{1}{2} + V, \frac{1}{(1 + \beta)\phi}\right)\) is either an unstable node or an unstable focus, and
\( (c^{FC}, k^{FC}) = \left( \frac{1}{2} - V; \frac{1}{(1 + \beta)\sigma} \right) \) is a saddle-point steady state.

At \( \phi = \sqrt{32\rho/(1 + \beta)} \), a so-called “saddle-node bifurcation” occurs as the last two steady states coincide and form a “semi-stable” steady state.

If \( \phi < \sqrt{32\rho/(1 + \beta)} \), the state-control system with positive production has one single steady state: \( (c^{FC}, k^{FC}) = (0, 0) \), which is unstable.

\begin{proof}
See Appendix K.
\end{proof}

5 Market Collusion and Incentives to Innovate: Full Cartel and R&D cartel confronted

The graphs showing different kinds of industry dynamics, each corresponding to one of the five distinct regions of the bifurcation diagram, are for the Full cartel qualitatively the same as the ones for the R&D cartel.\(^{29}\) The bifurcation diagram for the R&D cartel is together with its full cartel counterpart depicted in Figure 5.

Observe that the bifurcation curves for the full cartel lie below those for the R&D cartel, and this so over the entire parameter space. The relatively lower position of the shaded region (bounded above by the saddle-node and indifference-attractor bifurcation curves) for the full cartel leads to our first numerical observation from the global comparisons of the two regimes.

**Numerical observation 1.** For a given discount rate, it takes a larger market size and/or efficiency of R&D to make staying in a market a possibility for firms in the R&D cartel than it does for firms in the full cartel.

This observation stems from the facts that in the shaded region the exit trajectory is the only solution, while the threshold values characterize the space outside the shaded region. What is especially interesting to notice is the relative position of the curves indicating the point of indifference being exactly at \( c = 1 \). The curve for the full cartel lying below its R&D cartel counterpart implies the existence of the situation in which, for a given parameter combination, the full cartel regime has an indifference point at \( c > 1 \), while in the R&D cartel regime the indifference point is at some \( c < 1 \).\(^{30}\) This relates to our second observation.

**Numerical observation 2.** Whenever there is a threshold value of initial marginal costs in both regimes, this value is for the full cartel strictly larger than for the R&D cartel, for every point in the parameter space.

\(^{29}\)Remark 1, Lemmas 3 - 6, and Corollary 1 apply also to the Full cartel. Corresponding proofs are analogous to their R&D cartel counterparts. Any differences between the proofs are indicated within appendices corresponding to the R&D cartel.

\(^{30}\)In fact, this is the case for all points that in the bifurcation diagram lie between the two curves.
The above observation implies that, for a given market size, efficiency of R&D, and discount rate, the full cartel is ex ante more likely to bring a particular technology to a production phase and less likely to exit the market. This fact is further discerned by our next two numerical observations.

**Numerical observation 3.** The set of initial marginal costs that will induce the formation of a new market is larger if besides cooperating in R&D, firms also cooperate in the product market.

![Bifurcation diagram](image)

**Figure 5: Bifurcation diagram.** The bifurcation curves for the R&D cartel (grey) are drawn together with the bifurcation curves for the full cartel (black). Indicated regions correspond to the R&D cartel.

We already noted that the point of indifference in the region with $c \geq 1$ is an important concept. It determines the highest value of the initial marginal cost for which the firms still initiate investment. If the firms find out that the initial marginal cost (reflecting the starting level of the technology) is higher than this upper bound, they do not go into development and the particular technological idea is left unrealized - the market is not formed.

Figure 6 shows that this indifference point is for the full cartel at the higher value of the initial marginal cost ($\hat{c}_2$) than for the R&D cartel ($\hat{c}_1$). Hence, tech-

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31The threshold point is either a repeller or an indifference point. As the repelling point coincides with the unstable steady state $S_3$, its lower value in case of the R&D cartel follows simply from comparing the two corresponding steady-state values. The relative position of the indifference point is evaluated numerically (see also Figure 7).
technologies corresponding to the marginal costs between the two indifference points in the interval of a nontrivial size \((\hat{c}_1, \hat{c}_2)\) are the ones that are developed only if besides cooperating in R&D, firms also cooperate in the product market. The intuition is as follows. The firms are prepared to incur initial losses only if they know they will make large enough profits to compensate for these losses once the production has been initiated. The higher the value of the initial marginal cost, the more investments it takes to reduce the marginal cost to the level where the production can start (below the choke price) and so the larger the initial losses that have to be compensated for. As the firms in the full cartel are able to extract larger profits from the product market than the firms in the R&D cartel (which earn only competitive profits), the firms engaged in the full cartel can compensate for larger initial losses than the firms cooperating in R&D only.\(^{32}\)

As Figure 6 shows, the total discounted profit of the R&D cartel is for all initial technology levels \(c_0 \in (\hat{c}_1, \hat{c}_2)\) zero as firms optimally don’t do anything.\(^{33}\) However, the total profit of the full cartel is positive. Hence, the full collusion appears endogenously as a voluntary agreement through which both firms expect to benefit. Moreover, as the plots in the figure show, this agreement is beneficial also from the social standpoint as without it, consumers, as well as firms, are left empty-handed.\(^{34}\) Indeed, total discounted profit, consumer surplus, and total surplus are all zero for the R&D cartel when \(c_0 \geq \hat{c}_1\)! This is our next observation.

**Numerical observation 4.** Whenever both regimes have an indifference point in the region with zero production, there exists an interval of initial marginal costs for which the full cartel outperforms the R&D cartel with respect to consumer surplus and total surplus. The size of this interval increases with market demand and efficiency of the R&D process; the more so, the lower the discount rate.

Figure 7 illustrates this observation further by comparing the position of the indifference point for different parameter combinations. The indifference point is for the R&D cartel always below the one for the full cartel. For a given discount rate, increasing the market size and/or the efficiency of R&D, increases the value of the indifference point and so broadens the range of initial marginal costs that lead firms to select the stable path. It also increases the discrepancy between the two regimes \((\Delta_2 \hat{c} > \Delta_1 \hat{c})\); the more so, the lower the discount rate (and so the higher the slope of the convex curves). This is intuitive. The surplus created by R&D that can be appropriated by the firms in the future (once the production has started) is for the full cartel larger than for the R&D cartel. And the lower the discount rate, the higher the value of this future surplus for firms.

\(^{32}\)The assumption that firms know the market demand before the production stage has even started may seem strong. We could relax it by postulating that firms form expectations about future demand. However, while adding to the complexity of the model, this would not change our conclusions as long as these expectations are assumed to be identical between the two regimes; and we can not think of any good reason for not assuming so.

\(^{33}\)If the firms started developing the product, they would in total make a loss.

\(^{34}\)We assume that consumers discount the future at the same constant rate \(\rho > 0\) as the firms do.
Figure 6: State-control space (a), total discounted profit (b), consumer surplus (c), and total surplus (d), when the indifference point is in the region with zero production. The plots are drawn for parameter values \((\beta, \tilde{\rho}, \phi) = (1, 0.1, 2.25)\). Grey curves correspond to the R&D cartel, whereas the black ones correspond to the full cartel.

Figure 7 also shows that increasing the discount rate, for a given market size and efficiency of R&D, pushes the indifference point downwards (from \(C_1\) to \(C_2\)) as it reduces the value of the future profits. This increase in the discount rate can in principle be offset by sufficiently increasing the market size and/or the efficiency of R&D, as indicated in the figure.

In Figure 6, the R&D cartel outperforms the full cartel with respect to total and consumer surplus whenever both regimes lead to a new market. In Figure 8, we show that this is not a general fact. Indeed, for certain parameter combinations, the full cartel does, for some subset of initial marginal costs, lead to a higher consumer and total surplus even when both regimes bring the technology to the production stage. The relevant interval of initial marginal costs for which the full cartel brings about a higher total surplus is in such a case not \((\hat{c}_1, \hat{c}_2)\), but a wider \((c^*, \hat{c}_2)\).\(^{35}\) This occurs only when the starting technology levels which subsequently do lead to the

\(^{35}\)For the consumer surplus (graphs omitted for brevity), \(c^* \approx 3.9\) in the case considered.
Figure 7: Dependence of the indifference point \( \hat{c} \) on model parameters. Curves are drawn for three fixed values of \( \tilde{\rho} \). Curves for the R&D cartel (dotted) lie below the curves for the full cartel (full).

production stage are sufficiently high. The reason is that the full cartel invests relatively more than the R&D cartel and so also reaches the production stage relatively faster. Hence, positive profits and consumer surplus appear sooner as well. The higher the discount rate, the more noteworthy any time difference in reaching the production stage is as future surpluses are discounted more heavily. On the contrary, the lower the discount rate, the higher the value of \( c^* \) when it exists with a low discount rate it takes a comparatively high value of initial marginal cost for the difference between the regimes to manifest itself.\(^{36}\)

In Figure 6, we showed that for \( c_0 \in (\hat{c}_1, \hat{c}_2) \) the technology is developed only by the full cartel. Assuming \( c_0 = 2 \) for this particular parameter combination (observe that \( 2 \in (\hat{c}_1, \hat{c}_2) \)), Figure 9 shows how marginal cost and price are continuously decreasing over time to their long-run steady-state levels (0.0412 and 0.5206, respectively). We also plot the so-called Lerner Index which measures a divergence of price from marginal cost and is traditionally used as a short-cut estimator of the monopoly power and, as prevalently claimed, its corresponding

\(^{36}\)In figure 8, we have a relatively high discount rate \( \tilde{\rho} = 10 \), for instance, corresponding to \( \delta = 0.01 \) and \( \rho = 0.1 \) (recall that \( \tilde{\rho} = \rho/\delta \)). Figure 8 shows that in this case \( c^* \approx 3.6 \) for the total surplus. The lower the discount rate, the higher the value of \( c^* \) when it exists. For instance, for a lower discount rate \( \tilde{\rho} = 1 \) (corresponding to \( \delta = \rho \), \( c^* \) does not occur below the value of \( 1 \times 10^3 \), and so appears to be much less relevant a possibility.
Figure 8: Total surplus when the indifference point is in the region with zero production. The plots are drawn for parameter values \((\beta, \tilde{\rho}, \phi) = (1, 10, 50)\). Grey curves correspond to the R&D cartel, whereas the black ones correspond to the full cartel. \(c^* \approx 3.6, \hat{c}_1 \approx 4.01, \hat{c}_2 \approx 4.74\).

welfare loss. Its value is increasing over time to the “high” long-run level of 0.92. This increasing discrepancy between price and marginal cost is a consequence of the cartel increasing its mark-up over time to compensate for investments made before the production stage as well as to cover its current R&D expenses. This is clear from the plot of the instantaneous profit against the total discounted profit in Figure 9.

Figure 9: Marginal cost, price, and Lerner Index (a); total discounted profit and instantaneous profit (b), for the full cartel. The plots are drawn for parameter values \((\beta, \tilde{\rho}, \phi) = (1, 0.1, 2.25)\) and starting point \(c_0 = 2\).

Observe that it is only after a while that the instantaneous profit becomes positive. Its potential high size combined with a high Lerner Index may raise eyebrows of antitrust economists worrying about resource misallocation. We should note two things. First, while the instantaneous profit could well be judged high, focusing on
it, one overlooks huge R&D expenditures made over years (let alone all the mental effort). Notably, the total discounted profit is much lower: 0.0131 in our case (see Figure 9). In fact, the latter will be closer to zero, the closer to the indifference point the initial marginal cost is. Second, at Figure 7 we saw how market characteristics that affect future production (and so profitability) affect the position of the indifference point. Hence, if market regulations reduce mark-ups (for instance, by dismantling the product market cartel), the range of initial technologies that are developed further shrinks ($c_0^2$ ↓). It can then well happen that in our case $c_0(= 2) > \hat{c}_2$ and the firms are not able to cover the costs already incurred in R&D. One should note that in our example, the full cartel created the market and did not just take over the preexisting one. Had firms not been able to cooperate also in the product market, the market would not even have been formed (recall that $2 > \hat{c}_1$). Hence, the Lerner Index appears misleading from a correct dynamic perspective as it clearly cannot indicate “social inefficiencies” in any meaningful way. As shown, (larger) deviations of price from marginal costs are in our case not a source of inefficiencies but a requirement for technology to be developed and a market to be formed at all.\footnote{In the literature, the Lerner Index has been subject to several critiques. Pindyck (1985) claims that characteristics of dynamic markets (e.g., technological change, innovation, learning by doing) cause the Lerner Index to give a distorted picture of a firm’s monopoly power. The idea that deviations of price from marginal cost may arise from the need to cover incurred investment costs can be traced back to Lindenberg and Ross (1981):“...[The Lerner Index] does not recognize that some of the deviation of $P$ from $MC$ comes from ... the need to cover fixed costs and does not contribute to market value in excess of replacement cost” (p. 28). The argument that marginal pricing may be neither feasible nor desirable in technology-driven industries has been occasionally recognized by courts, e.g., in US v. Eastman Kodak (1995), where the court concludes that “Kodak’s film business is subject to enormous expenses that are not reflected in its short-run marginal costs”. See Elzinga and Mills (2011) for a comprehensive survey of limitations and qualifications of the index. For a critical discussion of numerous antitrust cases in business history, see Armentano (1999).}

**Numerical observation 5.** The set of initial marginal costs that induce firms to abandon the technology in time is larger if firms cannot collude in the product market.

The above observation, corresponding to the indifference point in the region with positive production, is illustrated in Figure 10. For all initial marginal costs in interval ($\hat{c}_1, \hat{c}_2$), the firms engaged in the R&D cartel gradually exit the market, whereas the firms engaged in the full cartel continue to reduce marginal costs and stay in the market. The revenue the firms in the R&D cartel can potentially earn in the product market is lower than that of the full cartel. Consequently, the relatively large expenses needed to stay in the market (to countervail depreciation) become for the R&D cartel “too high” sooner than for the full cartel. That is, the full cartel is more resistant to strained investment circumstances.\footnote{Strained in the sense of a relatively high depreciation rate in comparison with the market size and efficiency of R&D.}
Figure 10: State-control space (a), total discounted profit (b), consumer surplus (c), and total surplus (d), when the indifference point is within the region with positive production. The plots are drawn for parameter values $(\beta, \hat{\rho}, \phi) = (1, 0.1, 2)$. Grey curves correspond to the R&D cartel, whereas the black ones correspond to the full cartel. Full curves correspond to the stable path, whereas the dotted ones to the exit trajectory. Dots indicate the saddle-point steady state.

**Numerical observation 6.** When the stable path to the saddle-point steady state is optimal in both regimes, the steady-state value of marginal cost is for the full cartel lower than for the R&D cartel. The full cartel always invests more in R&D, be it along the stable path or the exit trajectory.

The fact that for every given marginal cost the trajectory for the full cartel is characterized by a higher R&D effort (see Figure 6, 10, and 11) underlies our next observation.

**Numerical observation 7.** For a given initial value of marginal cost, firms in the full cartel enter production phase earlier than firms in the R&D cartel. On the contrary, the speed at which firms abandon non-prospective technologies is lower when firms also collude in the product market than it is when they do not.

According to Observation 6, the full cartel always achieves higher production
efficiency than the R&D cartel when they both follow the stable path. However, when both regimes follow the stable path, the R&D cartel, for the very most part, brings about higher consumer and total surplus than the full cartel as the higher efficiency brought about by the full cartel is not relatively higher enough to compensate for the inefficiency arising by its relatively lower output, and it therefore does not translate into either higher consumer surplus or total surplus. Likewise, when both regimes are about to exit the market, the R&D cartel brings about higher total and consumer surplus. While the full cartel indeed abandon production relatively more slowly (recall Observation 7), this does not outweigh its relatively lower instantaneous production. Consequently, both consumer surplus and total surplus are lower for the full cartel (see also Figure 11).

However, as Figure 10 shows, the full cartel brings about a higher total surplus and also consumer surplus for initial marginal costs in interval \((\hat{c}_1, \hat{c}_2)\). Recall that for these costs, the R&D cartel eventually exits the market, while the full cartel follows the stable path and so stays in the market. Continuous reductions in marginal costs along the stable path by the full cartel eventually translate into a relatively higher output and in the end also into a higher total discounted consumer surplus and total surplus. These benefits of the full cartel, accruing from increasing efficiency over time, diminish with increases in the discount rate. As Figure 12 indicates, for a relatively higher discount rate, the R&D cartel can (for some subset of \((\hat{c}_1, \hat{c}_2)\)) lead to a higher total surplus when market size and/or R&D efficiency are sufficiently small even though the firms engaged in it are destined to exit the market. With a lower market size and efficiency of R&D, the full cartel itself invests less along the stable path and thereby brings about lower gains in efficiency; and this future gains are of a lower value, the higher the discount rate.

6 Discussion and concluding remarks

Significant benefits of R&D cooperations proven both in theory and practice have led policy makers not to treat such initiatives as per se violations of the antitrust law.

---

39 This follows directly from comparing the steady-state value of the marginal cost of the full cartel with that of the R&D cartel. The reason for this difference is that while both regimes indeed exert the same R&D effort in the steady state (the level needed to countervail depreciation), the investment paths of regimes are different. The full cartel invests more in R&D as it is able to appropriate a higher surplus resulting from it. This result stands in stark contrast to a strand of literature which claims that a lack of competition in the product market will through a lack of investment motivation lead firms to rest content with old methods and be engaged in x-inefficiency (a term first introduced by Leibenstein (1966)).

40 It is only for a relatively high discount rate (\(\hat{\rho} \gg 1\)) that the full cartel can dominate the R&D cartel in terms of total surplus when both regimes follow the stable path within Region II. The length of the interval of initial marginal costs for which this is the case appears negligible (e.g., for \(\rho = 20, \beta = 1, \phi = 30\), we have \(c^\ast \approx 0.960\), while \(\hat{c}_1 \approx 0.967\) and \(\hat{c}_2 \approx 0.985\)).

41 The intuition for this is that, having a monopoly power, the full cartel has still more to extract from the market and therefore has incentives to leave slower.

42 This result appears numerically robust over a wide range of the parameter space.
Figure 11: State-control space (a), total discounted profit (b), consumer surplus (c), and total surplus (d), when the exit trajectory is an optimal solution. The plots are drawn for parameter values $(\beta, \tilde{p}, \phi) = (1, 1, 2)$. Grey curves correspond to the R&D cartel, whereas the black ones correspond to the full cartel.

(as it holds for price-fixing agreements) but to consider them on a rule-of-reason basis. A possible extension of cooperation in R&D to that in the product market has been, however, widely considered as something that must be prevented at all costs.

Indeed, in their seminal paper, d’Aspremont and Jacquemin (1988) conclude, within their static model of R&D, that while the amount of R&D in the case of collusion in both R&D and product market is higher than in the case of R&D cooperation only, the quantity produced is still lower. Petit and Tolwinski (1999), who consider a dynamic model of R&D, conclude the same. Moreover, a higher price under full collusion leads in their model to a reduction in a consumer surplus and to the lowest total surplus among regimes considered. Consequently, the authors conclude that “...[full collusion] is socially inferior to other forms of industrial structures” and “...thus, antitrust legislation is important.” (p. 206). These conclusions seem to represent well the predominantly negative position of the received literature against market collusion.

Our global analysis however shows that conclusions of the received literature
Figure 12: Total surplus for the R&D cartel (grey curves) and full cartel (black curves) for two different combinations of parameter values: \((\beta, \tilde{\rho}, \phi) = (1, 1, 4)\) in the left figure and \((\beta, \tilde{\rho}, \phi) = (1, 1, 3.14)\) in the right figure. Full curves correspond to the stable path, whereas the dotted ones to the exit trajectory. Dots indicate the saddle-point steady state.

suffer from neglecting the very possibility that initial marginal costs are higher than the choke price. This makes existing models of R&D difficult to reconcile with empirical aspects of R&D. These include the observation that many initial technologies (prototypes or ideas) need to be developed further before they can enter the market, that only a minority of initial technologies is successfully brought to the market, and that production starts only after an initial stage during which the technology is developed further (we have warned of this already in Hinloopen, Smrkolj and Wagener (2011)).

Furthermore, by assuming that a particular product is already in the production stage and this so irrespective of the market regime under consideration, this literature disregards an important part of the development phase. It assumes an existence of a situation whose formation the true theory of R&D should in fact explain. The practice shows that firms do not pursue further all conceived technological ideas at all, but rest their decision upon the (expected) profitability of the latter. Yet, this profitability is affected not only by the (expected) demand for a product but also by the current and (expected) future industry structure.

Our analysis calls for a critical reassessment of competition policies which reduce mark-ups in products markets. With diminished future profits, the profitability of developing further any initial technology is lower. This fact is vital especially for technologies that come with marginal costs above the choke price. Not developing further these technologies does not surface as a cost of competition policy.

\footnote{Elmer Bolton, a scientist-manager at the DuPont company, one of the most innovative corporations in American business history, was famous for saying to company’s chemists who in his opinion lacked the awareness that the success of the company depends on its products being commercially exploitable: “This is very interesting chemistry, but somehow I don’t hear the tinkle of the cash register.” (Hounshell and Smith, 1988).}
as there is no visible production yet that will be taken from the market. We have shown that collusion in the product market through a larger profit enlarges the set of potential initial marginal costs for which the firms pursue further development of corresponding technologies. Preventing collaboration in the product market reduces this set.\textsuperscript{44} The potential loss of the total surplus this reduction brings about is then a hidden cost of competition policies. Yet, these indirect costs should obviously be taken into account when assessing properly the trade-off between static and dynamic efficiency. Moreover, our analysis indicates that the full cartel with its larger investments, and consequent faster innovations, might be a preferred option even when both regimes are expected to bring some technology to production if this technology calls for expensive investments and there is a strong preference for early production. Furthermore, in markets where demand is relatively low but firms need to continuously exert a lot of research effort to stay efficient and retain consumers, cooperation in the product market might be desirable as otherwise lower product market profits might induce firms to better exit the market.

We have also shown that the popular Lerner Index gives a distorted picture of welfare losses from a dynamic market perspective as relatively large deviations of price from marginal cost might in fact be needed for the market to be formed at all.

The practice of antitrust prosecutions however shows that antitrust authorities widely neglect the exposed problem. As reported by Armentano (1999) in his comprehensive exposition of past antitrust cases, “...price-fixing agreements were always illegal and there was to be no rule of reason with regard to such conspiracies” (p. 143) and “...whether prices have actually been fixed, or whether they have been fixed at unreasonable levels, has been \textit{immaterial} [at a federal court]” (p. 147).

In this paper we have focused on two market regimes (full cartel and R&D cartel), leaving aside, in particular, a regime where firms compete both on the product market and in their R&D endeavors. This latter option, which is clearly needed for a complete evaluation of competition policies, is part of our future research agenda.

References


\textsuperscript{44}Notice that the probability of being detected by the antitrust authorities when engaged in collusion can be accounted for as an increase in a firm’s discount rate (as it is standard in the literature). We have shown in Figure 7 that a higher discount rate reduces the value of marginal cost at the indifference point.


Appendices

A Proof of Lemma 1

A rescaled variable or parameter is distinguished by a tilde: for instance, if \( \pi \) denotes profit, then \( \tilde{\pi} \) denotes profit in rescaled variables. The profit function in the original (non-rescaled) model is:

\[
\pi_i = (A - q_i - q_j - c_i) q_i - bk_i^2
\]

Using the conversion rules given in Lemma 1, we obtain:

\[
\pi_i = (A - q_i - q_j - c_i) q_i - bk_i^2 = (A - A\tilde{q}_i - A\tilde{q}_j - A\tilde{c}_i) A\tilde{q}_i - b \left( \frac{A}{\sqrt{b}} \tilde{k}_i \right)^2 = A^2 \left( (1 - \tilde{q}_i - \tilde{q}_j - \tilde{c}_i) \tilde{q}_i - \tilde{k}_i^2 \right) = A^2 \tilde{\pi}_i
\]

The equation for the evolution of marginal cost over time is in original variables given by:

\[
\dot{c}_i(t) = c_i(t) \left( -k_i(t) - \beta k_j(t) + \delta \right).
\]

Write \( c_i(t) = c_i \left( \frac{1}{\delta} \tilde{t} \right) \). Then,

\[
\frac{dc_i}{dt} = \frac{dc_i}{\delta \, d\tilde{t}} = \frac{dc_i}{dt} \frac{dt}{\delta \, d\tilde{t}} = \frac{\dot{c}_i}{\delta} = c_i \left( 1 - \left( \frac{k_i}{\delta} - \beta \frac{k_j}{\delta} \right) \phi \right).
\]

Setting \( k_i = \frac{A}{\sqrt{b}} \tilde{k}_i \) and \( k_j = \frac{A}{\sqrt{b}} \tilde{k}_j \), and substituting them in the previous equation, we obtain:

\[
\frac{dc_i}{dt} = c_i \left( 1 - \left( \tilde{k}_i + \beta \tilde{k}_j \right) \frac{A}{\phi \sqrt{b}} \right),
\]

It is now natural to introduce \( \phi = \frac{A}{\delta \sqrt{b}} \).

Note that if \( \tilde{c}_i = c_i / A \), then \( \dot{\tilde{c}}_i = \dot{c}_i / A \) and

\[
\dot{\tilde{c}}_i = \tilde{c}_i \left( 1 - \left( \tilde{k}_i + \beta \tilde{k}_j \right) \phi \right).
\]

Observe finally that if \( t = \frac{\tilde{t}}{\delta} \), then \( e^{-\rho \tilde{t}} = e^{-\rho t} \) if, and only if, \( \tilde{\rho} = \frac{\rho}{\delta} \). \( \square \)
B Derivation of the optimal solution

In this section, we derive state-control and state-costate systems which satisfy necessary conditions for an optimal solution to the optimization problem of the R&D cartel and the Full cartel, respectively, using Pontryagin’s Maximum Principle.

B.1 R&D Cartel

The current-value Pontryagin function is given by

\[ P(c, k, \lambda) = \begin{cases} 
\frac{1}{9}(1-c)^2 - k^2 + \lambda c (1 - (1 + \beta)\phi k) & \text{if } c < 1, \\
-k^2 + \lambda c (1 - (1 + \beta)\phi k) & \text{if } c \geq 1.
\end{cases} \] (35)

Pontryagin’s Maximum Principle states that if the pair \((c^*, k^*)\) is an optimal solution, then there exists a function \(\lambda(t)\) such that \(c^*, k^*, \lambda\) satisfy the following conditions:

1.) \(k^*\) maximizes the function \(P\) for each \(t\):

\[ P(c^*, k^*, \lambda) = \max_{k \in \mathbb{R}^+} P(c^*, k, \lambda), \] (36)

2.) \(\lambda\) is a solution to the following co-state equation:

\[ -\frac{\partial P}{\partial c} = \dot{\lambda} - \rho \lambda \] (37)

which is evaluated along with the equation for the marginal cost

\[ \dot{c} = c(1 - (1 + \beta)\phi k) \] (38)

and the initial condition \(c(0) = c_0\).

Let \(k = K(c, \lambda)\) solve the problem \(\max_k P(c, k, \lambda)\) for every \((c, \lambda)\). We define the current-value Hamilton function

\[ H(c, \lambda) = P(c, K(c, \lambda), \lambda). \] (39)

The above necessary conditions (36)-(37) for an optimal solution are complemented by the transversality conditions\(^{45}\):

\[ \lim_{t \to \infty} e^{-\rho t} H(c, \lambda) = 0 \] (40)

and

\[ \lim_{t \to \infty} e^{-\rho t} \lambda c = 0. \] (41)

\(^{45}\)The necessity of (40), which allows exclusion of non-optimal trajectories, was proven by Michel (1982). For the necessity of transversality condition (41), see Kamihigashi (2001).
If problem (36) has a solution, this solution necessarily satisfies the following Karush-Kuhn-Tucker condition:

\[ \frac{\partial P}{\partial k} = -2k - (1 + \beta)\phi\lambda c \leq 0, \quad k \frac{\partial P}{\partial k} = 0. \]  

(42)

Conditions (42) imply that either i) \( k = 0 \) and \( \frac{\partial P}{\partial k} \leq 0 \) (implying \( \lambda \geq 0 \)) or ii) \( k > 0 \) and \( \frac{\partial P}{\partial k} = 0 \). In particular,

\[ k^* = \begin{cases} \frac{(1+\beta)\phi}{2} \lambda c & \text{if } \lambda \leq 0, \\ 0 & \text{if } \lambda > 0. \end{cases} \]  

(43)

The two-part composition of the Pontryagin function leads us to differentiate between two regimes of the state-costate system. The first one (in the region \( c < 1 \)) is characterized by positive production, whereas the second one (in the region \( c \geq 1 \)) corresponds to no production.

1. The system with positive production. In the region where \( c < 1 \), using (37), (38), and (43), we obtain the following optimal state and costate dynamics:

\[ \dot{c} = c \left( 1 + \frac{1}{2}(1 + \beta)\phi^2 \lambda c \right) \]
\[ \dot{\lambda} = \frac{2}{\phi} \left( 1 - c \right) + \left( \rho - 1 - \frac{1}{2}(1 + \beta)\phi^2 \lambda c \right) \lambda, \]

when \( \lambda \leq 0 \), and

\[ \dot{c} = c \]
\[ \dot{\lambda} = \frac{2}{\phi} \left( 1 - c \right) + \left( \rho - 1 \right) \lambda, \]

when \( \lambda > 0 \).

Equation (43) depends on the costate and state variable. Note that the correspondence between \( k \) and \( \lambda \) is one-to-one if, and only if, \( \lambda \leq 0 \), and that therefore in that case the state-costate system can be rewritten in the state-control form.

Differentiate equation (43) with respect to time to obtain the dynamic equation

\[ \frac{dk}{dt} \equiv \dot{k} = \frac{(1 + \beta)\phi}{2} \left( \dot{\lambda}c + \lambda \dot{c} \right). \]  

(46)

Substitute \( \dot{\lambda} \) from (37) and \( \dot{c} \) from (38). Note that equation (43) also implies that

\[ \lambda = - \frac{2k}{(1 + \beta)\phi c}. \]  

(47)

Substitute this expression into (46) to obtain:

\[ \dot{k} = \rho k - \frac{(1 + \beta)\phi}{9} c(1 - c). \]  

(48)

Hence, the system with positive production \( (c < 1) \) in the state-control form consists of the following two differential equations:

\[ \begin{cases} \dot{k} = \rho k - \frac{(1 + \beta)\phi}{9} c(1 - c), \\ \dot{c} = c \left( 1 - (1 + \beta)\phi k \right), \end{cases} \]  

(49)
where $k \geq 0$.

2. **The system with zero production.** In the region where $c \geq 1$, using (37), (38), and (43), we obtain the following optimal state and costate dynamics:

\[
\begin{align*}
\dot{c} &= c \left(1 + \frac{1}{2}(1 + \beta)^2 \phi^2 \lambda c\right), \\
\dot{\lambda} &= (\rho - 1 - \frac{1}{2}(1 + \beta)^2 \phi^2 \lambda c) \lambda, \\
\end{align*}
\]

(50)

when $\lambda \leq 0$, and

\[
\begin{align*}
\dot{c} &= c, \\
\dot{\lambda} &= (\rho - 1) \lambda,
\end{align*}
\]

(51)

when $\lambda > 0$.

Again, for $\lambda \leq 0$, the state-costate system can be rewritten in the state-control form:

\[
\begin{align*}
\dot{k} &= \rho k, \\
\dot{c} &= c \left(1 - (1 + \beta) \phi k\right),
\end{align*}
\]

(52)

where $k \geq 0$.

The Hamilton function, obtained by substituting (43) into the Pontryagin function (35), is given by

\[
H(c, \lambda) = \begin{cases} 
\frac{1}{9}(1 - c)^2 + \frac{1}{4}(1 + \beta)^2 \phi^2 \lambda^2 c^2 + \lambda c, & 0 \leq c < 1, \lambda \leq 0 \\
\frac{1}{9}(1 - c)^2 + \lambda c, & 0 \leq c < 1, \lambda > 0 \\
\frac{1}{4}(1 + \beta)^2 \phi^2 \lambda^2 c^2 + \lambda c, & c \geq 1, \lambda \leq 0 \\
\lambda c, & c \geq 1, \lambda > 0
\end{cases}
\]

(53)

The state-control analogue, valid for $\lambda \leq 0$, is given by

\[
H(c, k) = \begin{cases} 
\frac{1}{9}(1 - c)^2 + k \left(k - \frac{2}{(1+\beta)\phi}\right) & \text{if } c \in [0, 1) \\
k \left(k - \frac{2}{(1+\beta)\phi}\right) & \text{if } c \in [1, \infty),
\end{cases}
\]

(54)

where $k \geq 0$. The Hamilton function is of particular use when several paths satisfy the necessary conditions (those given by the Pontryagin’s Maximum Principle and transversality conditions) and we have to compare their corresponding values of the total discounted profit to determine which investment path is optimal. Indeed, if the pair $(c(\cdot), q(\cdot), k(\cdot))$ satisfies the necessary conditions (36)-(41), then

\[
\Pi(q, k, c) = \int_0^\infty \pi(q, k, c)e^{-\rho t} dt = \frac{1}{\rho} H(c(0), k(0)).
\]

(55)

In other words, to obtain the value of the total profit along any investment trajectory which satisfies the necessary conditions for an optimum, one only needs to evaluate the Hamilton function in its initial point (see, e.g., Grass et al. (2008), ch. 3, pp. 161–162).
B.2 Full Cartel

For the Full cartel, the current-value Pontryagin function is
\[ P(c, q, k, \lambda) = (1 - 2q - c) q - k^2 + \lambda c (1 - (1 + \beta) \phi k). \] (56)

Pontryagin’s Maximum Principle states that if the pair \((c^*, q^*, k^*)\) is an optimal solution, then there exists a function \(\lambda(t)\) such that \(c^*, q^*, k^*, \) and \(\lambda\) satisfy the following conditions:

1.) \(q^*\) and \(k^*\) maximize the function \(P\) for each \(t\):
\[ P(c^*, q^*, k^*, \lambda) = \max_{(q,k) \in \mathbb{R}^2_+} P(c, q, k, \lambda), \] (57)

2.) \(\lambda\) is a solution to the following co-state equation:
\[ -\frac{\partial P}{\partial c} = \dot{\lambda} - \rho \lambda \iff \dot{\lambda} = q + (\rho - 1 + (1 + \beta) \phi k) \lambda, \] (58)

which is evaluated along with the equation for the marginal cost
\[ \dot{c} = c(1 - (1 + \beta) \phi k) \] (59)

and the initial condition \(c(0) = c_0\).

Let \(Q(c, \lambda)\) and \(K(c, \lambda)\) solve the problem \(\max_{(q,k)} P(c, q, k, \lambda)\) for every \((c, \lambda)\). We define the current-value Hamilton function
\[ H(c, \lambda) = P(c, Q(c, \lambda), K(c, \lambda), \lambda). \] (60)

The above necessary conditions (57)-(58) for an optimal solution are complemented by the transversality conditions (40) and (41).

If problem (57) has a solution, this solution necessarily satisfies the following Karush-Kuhn-Tucker conditions:
\[ \frac{\partial P}{\partial q} = 1 - 4q - c \leq 0, \quad \frac{\partial P}{\partial q} = 0, \] (61)
\[ \frac{\partial P}{\partial k} = -2k - (1 + \beta) \phi \lambda c \leq 0, \quad k \frac{\partial P}{\partial k} = 0. \] (62)

Conditions (61) imply that either i) \(q = 0\) and \(c \geq 1\) or ii) \(q > 0\) and \(c < 1\). In particular,
\[ q^* = \begin{cases} (1 - c)/4 & \text{if } c < 1 \\ 0 & \text{if } c \geq 1 \end{cases} \] (63)

Conditions (62) imply that either i) \(k = 0\) and \(\frac{\partial P}{\partial k} \leq 0\) (implying \(\lambda \geq 0\)) or ii) \(k > 0\) and \(\frac{\partial P}{\partial k} = 0\). In particular,
\[ k^* = \begin{cases} -\frac{(1+\beta)\phi}{2} \lambda c & \text{if } \lambda \leq 0 \\ 0 & \text{if } \lambda > 0 \end{cases} \] (64)
The above conditions for the optimal production yields two regimes of the state-costate system. The first one is characterized by positive production, in the second one there is no production.

1. The system with positive production. In the region where $c < 1$, using (58), (59), (63), and (64), we obtain the following optimal state and costate dynamics:

\[
\begin{align*}
\dot{c} &= c \left( 1 + \frac{1}{2} (1 + \beta)^2 \phi^2 \lambda c \right) \\
\dot{\lambda} &= \frac{1}{4} (1 - c) + \left( \rho - 1 - \frac{1}{2} (1 + \beta)^2 \phi^2 \lambda c \right) \lambda, \\
\end{align*}
\]  

when $\lambda \leq 0$, and

\[
\begin{align*}
\dot{c} &= c \\
\dot{\lambda} &= \frac{1}{4} (1 - c) + (\rho - 1) \lambda, \\
\end{align*}
\]

when $\lambda > 0$.

We note again that the correspondence between $k$ and $\lambda$ in (64) is one-to-one if, and only if, $\lambda \leq 0$, and that therefore in that case the state-costate system can be rewritten in the state-control form:

\[
\begin{align*}
\dot{k} &= \rho k - \frac{(1 + \beta) \phi}{8} c(1 - c) \\
\dot{c} &= c (1 - (1 + \beta) \phi k), \\
\end{align*}
\]

where $k \geq 0$.

2. The system with zero production. In the region where $c \geq 1$, using (58), (59), (63), and (64), we obtain the following optimal state and costate dynamics:

\[
\begin{align*}
\dot{c} &= c \left( 1 + \frac{1}{2} (1 + \beta)^2 \phi^2 \lambda c \right) \\
\dot{\lambda} &= \left( \rho - 1 - \frac{1}{2} (1 + \beta)^2 \phi^2 \lambda c \right) \lambda, \\
\end{align*}
\]

when $\lambda \leq 0$, and

\[
\begin{align*}
\dot{c} &= c \\
\dot{\lambda} &= (\rho - 1) \lambda, \\
\end{align*}
\]

when $\lambda > 0$. Again, for $\lambda \leq 0$, the state-costate system can be rewritten in the state-control form:

\[
\begin{align*}
\dot{k} &= \rho k \\
\dot{c} &= c (1 - (1 + \beta) \phi k), \\
\end{align*}
\]

where $k \geq 0$.

The Hamilton function, obtained by substituting (64) and appropriate expressions for $q$ from (63) into the Pontryagin function (56), is given by

\[
H(c, \lambda) = \begin{cases} 
\frac{1}{8} (1 - c)^2 + \frac{1}{4} (1 + \beta)^2 \phi^2 \lambda^2 c^2 + \lambda c, & 0 \leq c < 1, \lambda \leq 0 \\
\frac{1}{8} (1 - c)^2 + \lambda c, & 0 \leq c < 1, \lambda > 0 \\
\frac{1}{4} (1 + \beta)^2 \phi^2 \lambda^2 c^2 + \lambda c, & c \geq 1, \lambda \leq 0 \\
\lambda c, & c \geq 1, \lambda > 0 
\end{cases}
\]
The state-control analogue, valid for $\lambda \leq 0$, is given by

$$
H(c, k) = \begin{cases} 
\frac{1}{2}(1 - c)^2 + k \left( k - \frac{2}{(1 + \beta)\phi} \right) & \text{if } c \in [0, 1), \\
\frac{k}{k - \frac{2}{(1 + \beta)\phi}} & \text{if } c \in [1, \infty), 
\end{cases}
$$

(72)

where $k \geq 0$.

C Proof of Lemma 7 and Lemma 2

As $\rho > 0$, the equilibrium condition $\dot{k} = \rho k = 0$ implies that $k = 0$. But then $\dot{c} = c (1 - (1 + \beta)\phi k) = c$. This is equal to 0 if, and only if, $c = 0$. But this cannot be as in the system with zero production we have $c \geq 1$.

D Proof of Proposition 1

Assume $\phi > 6\sqrt{\rho}/(1 + \beta)$. The stability of the steady-states can be analyzed by evaluating the trace and determinant of the following Jacobian matrix:\footnote{Note that the trace of the respective Jacobian matrix is equal to the sum of eigenvalues, while its determinant is equal to their product. If the real part of each eigenvalue is negative, then the steady state is asymptotically stable. If the real part of at least one of the eigenvalues is positive, then the steady state is unstable. More particular, if one eigenvalue is real and positive and the other one real and negative, the steady state is said to be a saddle. In this last situation, there are four special trajectories, the separatrices, two of which are forward asymptotic to the saddle, while the other two are backward asymptotic to it. The union the former separatrices with the saddle form the stable manifold of the saddle; analogously, the union of the latter with the saddle form the unstable manifold.}

$$
J^{RC} = \begin{bmatrix} \frac{\partial \dot{c}}{\partial c} & \frac{\partial \dot{c}}{\partial k} \\
\frac{\partial \dot{k}}{\partial c} & \frac{\partial \dot{k}}{\partial k} \end{bmatrix} = \begin{bmatrix} 1 - (1 + \beta)\phi k & -(1 + \beta)\phi c \\
-(1 + \beta)\phi(1 - 2c) & \rho \end{bmatrix}. 
$$

(73)

At $c = k = 0$, the trace $\tau$ of the matrix $J^{RC}$ is given as

$$
\tau \overset{\text{def}}{=} \text{tr} J^{RC} = 1 + \rho > 0,
$$

its determinant $\Delta$ is

$$
\Delta \overset{\text{def}}{=} \det J^{RC} = \rho > 0,
$$

and its discriminant $D$ is

$$
D \overset{\text{def}}{=} \tau^2 - 4\Delta = (1 - \rho)^2 > 0.
$$

Hence, this steady state is an unstable node.

Evaluating the Jacobian matrix at

$$
k = \frac{(1 + \beta)\phi}{9\rho} c(1 - c), \quad c = \frac{1}{2} + V = \frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{36\rho}{(1 + \beta)^2\phi^2}}.
$$

(74)
we obtain $\tau = \rho > 0$ and

\[
\Delta = \frac{\sqrt{(1 + \beta)^2 \phi^2 - 36\rho} \left( (1 + \beta)\phi + \sqrt{(1 + \beta)^2 \phi^2 - 36\rho} \right)}{18},
\]

which is clearly positive if $\phi > 6\sqrt{\rho}/(1 + \beta)$. Hence, this steady state is also unstable. The discriminant takes the value

\[
D = \rho(8 + \rho) - \frac{2}{9}(1 + \beta)\phi \left( (1 + \beta)\phi + \sqrt{(1 + \beta)^2 \phi^2 - 36\rho} \right),
\]

which is zero for $\phi = \phi_0 = \frac{3\sqrt{\rho(8 + \rho)^2}}{2\sqrt{(1 + \beta)^2(4 + \rho)}}$. The steady state is an unstable node if $D > 0$ and an unstable focus if $D < 0$. In the latter case, the eigenvalues of $J^{RC}$ are complex conjugates with positive real parts.\(^{47}\)

Finally, evaluating the Jacobian matrix at

\[
k = \frac{(1 + \beta)\phi}{9\rho}c(1 - c), \quad c = \frac{1}{2} - V = \frac{1}{2} - \frac{1}{2} \sqrt{1 - \frac{36\rho}{(1 + \beta)^2 \phi^2}},
\]

we obtain $\tau = \rho > 0$ and

\[
\Delta = \frac{(1 + \beta)^2 \phi^2 - 36\rho - (1 + \beta)\phi \sqrt{(1 + \beta)^2 \phi^2 - 36\rho}}{18}.
\]

If $\phi > 6\sqrt{\rho}/(1 + \beta)$, then $\Delta < 0$, and the eigenvalues are real and have opposite sign. Therefore, (78) is the saddle-point steady state of the system. Observe that for $\phi = 6\sqrt{\rho}/(1 + \beta)$, the two steady states (74) and (78) coincide at $c = 1/2$ and $k = \frac{1}{(1 + \beta)\phi}$. A saddle-node bifurcation occurs at these parameter values, where the two equilibria collide and disappear.\(^{48}\) Substituting the expression for the steady-state marginal cost of the steady states other than the origin into (20), the expression for the optimal investment in the steady state simplifies to $k^{RC} = \frac{1}{(1 + \beta)\phi}. \quad \Box$

### E Proof of Remark 1

The Hessian matrix of the Pontryagin function with respect to control and state variables is

\[
D^2_{(k,c)} P = \begin{bmatrix}
\frac{\partial^2 P(\cdot)}{\partial k^2} & \frac{\partial^2 P(\cdot)}{\partial c \partial k} \\
\frac{\partial^2 P(\cdot)}{\partial c \partial k} & \frac{\partial^2 P(\cdot)}{\partial c^2}
\end{bmatrix}
= \begin{bmatrix}
-2 & -(1 + \beta)\phi \lambda \\
-(1 + \beta)\phi \lambda & \frac{2}{9}
\end{bmatrix}
\]

\(^{47}\)Note that the eigenvalues of $J^{RC}$ are $r_{1,2} = \frac{1}{2} (\tau \pm \sqrt{\tau^2 - 4\Delta})$.

\(^{48}\)In Figure 1, this corresponds to the locus $\dot{c} = 0$, which is the horizontal line $k = \frac{1}{(1 + \beta)\phi}$, becoming tangent to the locus $\dot{k} = 0$ at its peak.
The determinant of the above matrix is \(-4/9 - (1 + \beta)^2 \phi^2 \lambda^2\), which is clearly negative. Notice that the determinant is the leading principal minor of the second order. As the leading principal minor of the even order is negative, the Hessian is by definition indefinite and, therefore, the Pontryagin function is nowhere jointly concave in state and control variables. Hence, the Arrow-Mangasarian sufficiency conditions are not satisfied.\(^\text{49}\)

For the Full cartel, the Hessian matrix of the Pontryagin function with respect to control and state variables is

\[
D^2_{(q,k,c)} P = \begin{bmatrix}
\frac{\partial^2 P(\cdot)}{\partial q^2} & \frac{\partial^2 P(\cdot)}{\partial q \partial k} & \frac{\partial^2 P(\cdot)}{\partial q \partial c} \\
\frac{\partial^2 P(\cdot)}{\partial k \partial q} & \frac{\partial^2 P(\cdot)}{\partial k^2} & \frac{\partial^2 P(\cdot)}{\partial k \partial c} \\
\frac{\partial^2 P(\cdot)}{\partial c \partial q} & \frac{\partial^2 P(\cdot)}{\partial c \partial k} & \frac{\partial^2 P(\cdot)}{\partial c^2}
\end{bmatrix} = \begin{bmatrix}
-4 & 0 & -1 \\
0 & -2 & -(1 + \beta)\phi\lambda \\
-1 & -(1 + \beta)\phi\lambda & 0
\end{bmatrix}
\]

(81)

The leading principal minor of the first order is \(-4\). The leading principal minor of the third order, which is also the determinant, of the above matrix is equal to \(2 + 4(1 + \beta)^2 \phi^2 \lambda^2\). As the leading principal minors of odd order do not have the same sign, the matrix is indefinite. Consequently, the Pontryagin function is nowhere jointly concave in state and control variables. Hence, the Arrow-Mangasarian sufficiency conditions are not satisfied.

\[\square\]

F Proof of Lemma 3

To prove the lemma, we consider the state-co-state form of the solutions, given by:

\[
\dot{c} = \begin{cases}
0 & 0 \leq c < 1, \lambda \leq 0 \\
c \left(1 + \frac{1}{2}(1 + \beta)^2 \phi^2 \lambda c\right) & 0 \leq c < 1, \lambda > 0 \\
c & c \geq 1, \lambda \leq 0 \\
c & c \geq 1, \lambda > 0
\end{cases}
\]

(82)

\[
\dot{\lambda} = \begin{cases}
\frac{2}{\tau}(1 - c) + (\rho - 1 - \frac{1}{2}(1 + \beta)^2 \phi^2 \lambda c)\lambda, & 0 \leq c < 1, \lambda \leq 0 \\
\frac{2}{\tau}(1 - c) + (\rho - 1)\lambda, & 0 \leq c < 1, \lambda > 0 \\
(\rho - 1 - \frac{1}{2}(1 + \beta)^2 \phi^2 \lambda c)\lambda, & c \geq 1, \lambda \leq 0 \\
(\rho - 1)\lambda, & c \geq 1, \lambda > 0
\end{cases}
\]

(83)

\[
H(c, \lambda) = \begin{cases}
\frac{1}{\tau}(1 - c)^2 + \frac{1}{4}(1 + \beta)^2 \phi^2 \lambda^2 c^2 + \lambda c, & 0 \leq c < 1, \lambda \leq 0 \\
\frac{1}{\tau}(1 - c)^2 + \lambda c, & 0 \leq c < 1, \lambda > 0 \\
\frac{1}{4}(1 + \beta)^2 \phi^2 \lambda^2 c^2 + \lambda c, & c \geq 1, \lambda \leq 0 \\
\lambda c, & c \geq 1, \lambda > 0
\end{cases}
\]

(84)

\[^{49}\text{For details of sufficiency conditions, see, for instance, Grass et al. (2008).}\]
where $\tau = 9$ for the R&D cartel and $\tau = 8$ for the Full cartel.

Introduce the characteristic function $\chi_S$ of a set $S$ by

$$
\chi_S(x) = \begin{cases} 
1 & \text{if } x \in S \\
0 & \text{if } x \notin S.
\end{cases}
$$

(85)

Differential equations (82) and (83) can be rewritten more compactly as:

$$
\dot{c} = c + \chi_{(-\infty, 0]}(\lambda) \frac{1}{2} (1 + \beta)^2 \phi^2 \lambda c^2 \overset{\text{def}}{=} F(c, \lambda)
$$

(86)

and

$$
\dot{\lambda} = (\rho - 1)\lambda - \chi_{(-\infty, 0]}(\lambda) \frac{1}{2} (1 + \beta)^2 \phi^2 c \lambda^2 + \chi_{[0, 1]}(c) \frac{2}{\tau} (1-c) \overset{\text{def}}{=} G(c, \lambda).
$$

(87)

The state-co-state space is sketched in Figure 13. We first consider the trajectories

![Diagram](image)

**Figure 13:** *Illustrative sketch of the state–co-state space (for the full cartel with $\rho < 1$).* The dotted vertical line $c = 1$ separates the region with zero production from the region with a positive level of production, whereas the horizontal line $\lambda = 0$ separates the region with positive investment from the region with zero investment. The loci $\dot{c} = 0$ and $\dot{\lambda} = 0$ intersect in the two unstable steady states ($S_1$ and $S_3$) and the saddle-point steady state $S_2$ (not indicated). A number of trajectories is indicated by black curves: the arrows point in the direction of the flow. The thick curve indicates the stable path leading to the saddle-point steady state.
in the region where \( c \geq 1 \) and \( \lambda > 0 \), which are the solution to the following canonical system:

\[
\begin{aligned}
\dot{c} &= c \\
\dot{\lambda} &= (\rho - 1)\lambda,
\end{aligned}
\]

given by \( \lambda(t) = C_1 e^{(\rho - 1)t} \) and \( c(t) = C_2 e^t \), where \( C_1 \) and \( C_2 \) are positive constants. As mentioned in the main text, every candidate for an optimal solution must necessarily satisfy the transversality condition \( \lim_{t \to \infty} e^{-\rho t} \lambda(t) c(t) = 0 \). As we now show, the trajectories in the region considered violate this condition. Moreover, they also violate the following transversality condition:

\[
\lim_{t \to \infty} e^{-\rho t} H(c, \lambda) = 0,
\]

which in this case coincides with the first transversality condition, as shown below. It follows from Michel (1982) that condition (89)\(^{50}\) is also a necessary condition, and hence that it allows exclusion of trajectories which verify the other necessary conditions in an infinite horizon optimization problem. The value of the Hamiltonian function evaluated along the considered trajectories is \( H(c, \lambda) = \lambda(t)c(t) = C_1 C_2 e^{\rho t} \); it follows that

\[
\lim_{t \to \infty} e^{-\rho t} H(c, \lambda) = \lim_{t \to \infty} e^{-\rho t} \lambda(t) c(t) = C_1 C_2 \neq 0.
\]

Hence, no trajectory in the region given by the restriction \( c \geq 1 \) and \( \lambda > 0 \) can be optimal.

Consider now trajectories in the region with \( 0 < c < 1 \) and \( \lambda > 0 \). These trajectories are the solution to

\[
\begin{aligned}
\dot{c} &= c \\
\dot{\lambda} &= \frac{2}{\tau}(1 - c) + (\rho - 1)\lambda.
\end{aligned}
\]

We show that any trajectory in this region sooner or later enters the region with \( c \geq 1 \) and \( \lambda > 0 \).

Observe that the equation \( \dot{c} = c \) has as a solution \( c(t) = C_3 e^t \), where \( C_3 \) is a positive constant. Hence, along any trajectory in this region, \( c \) is increasing. From (83) follows that if \( \lambda = 0 \), then

\[
\dot{\lambda} = \frac{2}{\tau}(1 - c) > 0
\]

for all \( c \in (0, 1) \). Hence, trajectories in the region with \( 0 < c < 1 \) and \( \lambda > 0 \) cannot exit this region through the line segment \( \{(c, \lambda) : c \in (0, 1), \lambda = 0\} \). We now show that they also cannot exit through the point \((c, \lambda) = (1, 0)\)\(^{51}\).

\(^{50}\)In words, the above condition means that the present value of the maximum of the Pontryagin function (the present value of the Hamiltonian) converges to zero when time goes to infinity.

\(^{51}\)If they exited through this point, they would satisfy the transversality conditions as \( \dot{\lambda} = 0 \) for \( \lambda = 0 \) and \( c \geq 1 \).
Let \( x = (c, \lambda) \) and \( x_0 = (c_0, \lambda_0) = (1, 0) \). Furthermore, let \( \vec{\mathfrak{f}} : D \rightarrow \mathbb{R}^2 \) be a vector function defined as

\[
\vec{\mathfrak{f}}(c, \lambda) = \left( F(c, \lambda), G(c, \lambda) \right),
\]

where \( F \) is defined in (86) and \( G \) in (87).\(^{52} \) Its domain is \( D = \mathbb{R}_+ \times \mathbb{R} \subset \mathbb{R}^2 \). We are then looking for a solution to a 2-dimensional nonlinear autonomous dynamical system of the form

\[
\dot{x} = \vec{\mathfrak{f}}(x), \quad x(0) = x_0.
\]  

(92)

Take \( 0 < r < 1 \). Then the set \( D_0 = B_r(x_0) = \{ x \in D : \| x - x_0 \| \leq r \} \), (93)

is a neighborhood of \( x_0 \). First, we show that the restriction of the functions \( F \) and \( G \) on \( D_0 \) (a compact subset of \( D \)) is Lipschitz\(^{53} \). Consider first the function

\[
F(c, \lambda) = c + \lambda \chi_{(-\infty, 0]}(\lambda) \left(1 + \beta \right)^2 \phi^2 \frac{c^2}{2}.
\]

Write for brevity \( \chi = \chi_{(-\infty, 0]} \) and set \( h(\lambda) = \chi(\lambda) \lambda \). Then,

\[
|h(\lambda) - h(0)| = |\chi(\lambda) \lambda - 0| = |\chi(\lambda)\lambda| \leq |\lambda| = 1 \cdot |\lambda - 0|.
\]

(94)

Hence, \( h \) is Lipschitz on \( D_0 \). As on compact sets continuously differentiable functions are Lipschitz, as well as sums and products of Lipschitz functions, it follows that \( F \) is Lipschitz on \( D_0 \).

Consider now on \( D_0 \) the function

\[
G = (\rho - 1) \lambda - \lambda^2 \chi_{(-\infty, 0]}(\lambda) \left(1 + \beta \right)^2 \phi^2 \frac{c^2}{2} + \frac{2(1 - c)}{\tau} \chi_{[0,1]}(c).
\]

The first term of this expression is a linear function; the second term is the function \( h \) introduced above times a differentiable function. That the final term is Lipschitz at \( c = 1 \) is demonstrated in the same way as for the function \( h \). It follows that \( G \) is Lipschitz on \( D_0 \).

We have proved that the functions \( F \) and \( G \) are locally Lipschitz at \( x_0 \). Consequently, so is also the vector function \( \vec{\mathfrak{f}} \). By the Picard-Lindelöf Theorem, the

\(^{52}\)Though continuous, functions \( F \) and \( G \) are not differentiable: \( F \) is not differentiable with respect to \( \lambda \) at \( \lambda = 0 \), whereas \( G \) is not differentiable with respect to \( c \) at \( c = 1 \). Using Peano’s existence theorem, the continuity of \( \vec{\mathfrak{f}} \) implies that at least one solution to (92) exists. However, continuity is not enough to guarantee uniqueness. An additional condition that needs to be fulfilled to guarantee uniqueness of solutions, at least in some neighborhood of \( x_0 \), is that \( \vec{\mathfrak{f}} \) is locally Lipschitz in \( x \) at \( x_0 \).

\(^{53}\)Simply put, a function \( f : D \rightarrow \mathbb{R}^n \) is said to be Lipschitz on \( B \subseteq D \) if there exists a constant \( K > 0 \) such that \( \| f(x) - f(y) \| \leq K \| x - y \| \), for all \( x, y \in B \). See Kelley and Peterson (2010), ch. 8, and Sohrab (2003), ch. 4, for the introduction to concepts and precise definitions of terms used in this section.
system (92) has a unique solution in a neighborhood of \( x_0 \). As this point is on the exit trajectory, no other trajectory can pass through it, in particular no trajectory from the region with \( \lambda > 0 \). As \( \dot{c} > 0 \) in the region with \( 0 < c < 1 \) and \( \lambda > 0 \), as shown above, all trajectories in this region exit through \( \{c = 1, \lambda > 0\} \) and enter the region with \( c \geq 1 \) and \( \lambda > 0 \) as \( t \to \infty \). We have already shown that none of these can be optimal.

Moreover, trajectories through points in the set \( \{(c, \lambda) : c \in (0, 1), \lambda = 0\} \) satisfy \( \dot{\lambda} > 0 \); they move in the region for which \( \lambda > 0 \) and hence cannot be a part of any optimal trajectory. Due to a one-to-one correspondence between \( \lambda \) and \( k \) as given in the first part of (64), this set corresponds to the set \( \{(c, k) : c \in (0, 1), k = 0\} \) in the state-control representation of the solution. As we have shown, all trajectories leading to these points violate the transversality conditions which every optimal trajectory must necessarily satisfy.

\[ G \quad \text{Proof of Lemma 4} \]

Assume that there is an optimal investment schedule for which \( c \to 0 \) and \( k \to \infty \) as \( t \to \infty \).

The instantaneous profit function, corresponding to the symmetric equilibrium, of each firm in every instant is:

\[ \pi = (1 - 2q - c)q - k^2. \tag{95} \]

As \( c \to 0 \) along the trajectories considered, they sooner or later enter the region with positive production, where we have \( q = (1-c)/3 \). Substituting this expression into (95), we obtain:

\[ \pi = \frac{(1-c)^2}{9} - k^2. \tag{96} \]

As \( c \to 0 \), \( (1-c)^2/9 \) approaches its upper bound of \( 1/9 \). The second term in the above equation, \( -k^2 \), decreases beyond all bounds as \( k \to \infty \). As \( t \to \infty \), there is therefore a time \( t_0 \) such that \( \pi(t) = 0 \) for \( t = t_0 \), and \( \pi(t) < 0 \) for all \( t > t_0 \). Changing the investment schedule to \( k(t) = 0 \) for all \( t \geq t_0 \) would yield a higher value of total discounted profits \( \Pi \), contradicting the assumption.

In the case of the Full cartel, the instantaneous profit is

\[ \pi(c, k) = \frac{(1-c)^2}{8} - k^2. \tag{97} \]

As \( c \to 0 \), \( (1-c)^2/8 \) approaches its upper bound of \( 1/8 \). The second term in the above equation, \( -k^2 \), however decreases beyond all bounds as \( k \to \infty \). The rest of the argument is analogous to the one for the R&D cartel above. \( \square \)
H Proof of Lemma 5

We know that the state-control system with zero production is:

\[
\begin{align*}
\dot{k}(t) &= \rho k(t) \\
\dot{c}(t) &= c(t) \left(1 - (1 + \beta)\phi k(t)\right).
\end{align*}
\]

At \( k = 0 \), this reduces to: \( \dot{k}(t) = 0, \dot{c}(t) = c \). Hence, marginal costs increase to infinity along the exit trajectory as \( t \to \infty \). However, as \( \lambda = 0 \) for \( c \geq 1 \) along the exit trajectory, the transversality condition (41) is satisfied. Observe that along the exit trajectory \( k > 0 \) for \( 0 < c < 1 \) and \( k = 0 \) for \( c \geq 1 \). Consequently, it follows from (43) (or (64) for the Full cartel) that \( \lambda = 0 \) at \( c = 1 \). Then, from (83) we have that indeed \( \lambda = 0 \) for all \( c \geq 1 \). Increasing marginal costs along the exit trajectory sooner or later exceed the value of 1, for which the value of \( H(c, \lambda) \) becomes 0 (see (84)). Hence, the transversality condition (40) is satisfied as well.

I Proof of Corollary 1

In this appendix, we prove that the only candidates for an optimal solution curve are the stable path of the saddle-point steady state and the exit trajectory. In particular, we show that any solution curve of the state-control system, given in (49) and (52) and depicted in Figure 1, starting at a point \((c_0, k_0)\) with \( c_0, k_0 > 0 \) either (i) ends on the stable path, (ii) ends on the exit trajectory, (iii) gives rise to a control \( k(t) \) that goes to infinity, and then satisfies the condition of Lemma 4, or (iv) passes through the line segment \( \{(c, k) : c \in (0, 1), k = 0\} \), and is then excluded as an optimal solution by Lemma 3. We note that the vector field defined by the state-control system is always locally Lipschitz, and that therefore the theorem of existence and uniqueness of trajectories through a given initial point holds.

There are two situations, determined by the location of the maximum \((\frac{1}{2}, k^*)\) of the \( \dot{k} = 0 \) isocline, which is the quadratic function \( c \mapsto (1 + \beta)\phi_36\rho c(1 - c) \). In the first situation \( k^* \geq \frac{1}{(1 + \beta)\phi} \), in the second \( k^* < \frac{1}{(1 + \beta)\phi} \).

If \( k^* \geq \frac{1}{(1 + \beta)\phi} \), the state-control space \( S = \{(c, k) : c > 0, k > 0\} \) can be partitioned in three regions \( S_1, S_2, \) and \( S_3 \). The first set is defined as follows:

\[
S_1 = \{(c, k) : 0 < c < 1, k > k^*\},
\]

where \( k^* = \frac{(1 + \beta)\phi}{36\rho} \) is the maximum of \( c \mapsto (1 + \beta)\phi_36\rho c(1 - c) \). To define the second set, we note that the trajectory \( \gamma \) of the state-control space that passes through the point \((c, k) = (1, (\frac{(1 + \beta)\phi}{36\rho})^*)\), when continued backwards in time, necessarily has a second intersection with the line \( c = 1 \). The first of these intersections, as time decreases, is denoted \((c, k) = (1, k_*), \) where \( 0 < k_* < \frac{1}{(1 + \beta)\phi} \). Let \( D \) be the region bounded by \( \gamma \) and the line \( c = 1 \). Then,

\[
S_2 = \{(c, k) : c \geq 1, k > 0\} \setminus D.
\]
Finally, 

\[ S_3 = S \setminus (S_1 \cup S_2). \]

From the state-control equations

\[ \dot{c} = c(1 - (1 + \beta)\phi k), \quad \dot{k} = \rho k - \frac{(1 + \beta)\phi}{9} c(1 - c), \]

and the fact that \( k^* > \frac{1}{(1+\beta)\phi} \), it follows that everywhere in \( S_1 \) we have \( \dot{c} < 0 \) and \( \dot{k} > 0 \). It follows by Lemma 4 that no trajectory in this region can be optimal.

In region \( S_2 \), the state-control equations read as

\[ \dot{c} = c(1 - (1 + \beta)\phi k), \quad \dot{k} = \rho k. \]

We claim that every trajectory in this region must leave it through the half-line \( \ell_1 = \{(c, k) : c = 1, k \geq k^*\} \). Note first that the boundary of \( S_2 \) consists of \( \ell_1 \), the curve \( \gamma \), the line segment \( \ell_2 = \{(c, k) : c = 1, 0 < k \leq k_4\} \), and the half-line \( \ell_3 = \{(c, k) : c \geq 1, k = 0\} \). As \( \gamma \) and \( \ell_3 \) are parts of trajectories of the state-control system, no trajectories can leave \( S_2 \) through them. Moreover, we have \( \dot{c} \geq 0 \) on \( \ell_2 \), so exit through this part of the boundary is also impossible. The remaining possibilities are to leave through \( \ell_1 \), as claimed, or to remain in \( S_2 \) indefinitely.

To show that the latter alternative is impossible, note that since \( \dot{k} = \rho k \), any trajectory in \( S_2 \) will eventually satisfy \( k > 2/(1+\beta)\phi \). But then \( \dot{c} < -c \), and this implies that eventually \( c \) should satisfy \( c = 1 \), leaving the region \( S_2 \) towards \( S_1 \). As no trajectory in \( S_1 \) can be optimal, this now extends to all trajectories in \( S_2 \).

It remains to analyze the trajectories in \( S_3 \). They can leave that region through the line segments \( \ell_4 = \{(c, k) : 0 < c < 1, k = 0\} \) or \( \ell_5 = \{(c, k) : 0 < c < 1, k = k^*\} \), as the remaining parts of the boundary are trajectories of the state-control system, or through the point \((1, 0)\) on the exit trajectory.

Trajectories leaving through \( \ell_4 \) enter \( S_2 \) and therefore cannot be optimal. As noted above, trajectories leaving through \( \ell_4 \) are excluded by Lemma 3 from optimality. Trajectories leaving through \( \ell_5 \) enter \( S_1 \) and again cannot be optimal. Of all the trajectories leaving \( S_3 \), only those on the exit trajectory are thus candidates for optimal solutions.

It remains to discuss the trajectories that remain in \( S_3 \) for all time. By the Poincaré-Bendixon theorem, since \( S_3 \) is bounded, the limit set of such a trajectory is either a steady-state point or a closed curve. The latter possibility can be ruled out as the area enclosed by the curve would be invariant (cf. Wagener, 2003). The only steady state in \( S_3 \) that can be approached by a trajectory is the saddle, if \( k^* > \frac{1}{(1+\beta)\phi} \), or the semi-stable steady state if \( k^* = \frac{1}{(1+\beta)\phi} \), and this shows the result.
If $k^* < \frac{1}{(1+\beta)\phi}$, the situation is much simpler. Define in that case

\[ S_1 = \{(c,k) : 0 < c < 1, k > \frac{1}{(1+\beta)\phi}\}, \]
\[ S_2 = \{(c,k) : c \geq 1, k > 0\}, \]
\[ S_3 = S \setminus (S_1 \cup S_2). \]

As above, points in $S_1$ cannot be optimal as a consequence of Lemma 4; points in $S_2$ eventually end up in $S_1$ and are excluded by the same reasoning; and as there are no saddle points in $S_3$ and the trajectories leaving through $\ell_4$ cannot be optimal, the only remaining candidate is the exit trajectory. The proof for the Full cartel is analogous.

### J Proof of Lemma 6

From (54), we know that the Hamiltonian for $c \geq 1$ is given by:

\[ H(c, k) = k \left( k - \frac{2}{(1+\beta)\phi} \right), \]

(98)

and from (55), we know that the comparison of the total discounted profits of each two candidate optimal paths amounts to comparing the values of the respective Hamiltonians in the initial point of each respective path. As the Hamiltonian in the case of zero investment and zero production is zero, the indifference point between the stable path and the exit trajectory in the region with zero production must be the point at which the Hamiltonian evaluated along the stable path obtains the value of zero.

\[ H(c, k) = k \left( k - \frac{2}{(1+\beta)\phi} \right) = 0 \]
\[ \Rightarrow k = 0 \quad \text{or} \quad k = \frac{2}{(1+\beta)\phi} \]

The solution structure (24)-(25) tells us that the stable path in the region with $c \geq 1$ decreases as $t \to -\infty$, assuming that the stable path covers the respective region. Observe that the derivative of the Hamiltonian with respect to $k$ is $\frac{\partial H(c,k)}{\partial k} = 2k - \frac{2}{(1+\beta)\phi}$, which is positive for $k > \frac{1}{(1+\beta)\phi}$ and negative for $k < \frac{1}{(1+\beta)\phi}$.

If the stable path enters the region with zero production at all, it enters this region at the point where $k > \frac{1}{(1+\beta)\phi}$; this follows directly from equation (52).

The same equation implies that $k$ is decreasing along the stable path as $t \to -\infty$, assuming the stable path enters the zero-production region. The above conclusions lead us to distinguish three cases. First, if the stable path crosses the boundary line $c = 1$ at $k > \frac{2}{(1+\beta)\phi}$, then the value of the Hamiltonian is decreasing along the stable path in the region with $c \geq 1$ as $t \to -\infty$, passes zero at $k = \frac{2}{(1+\beta)\phi}$ and is negative afterwards. In this case, the indifference point is the value of the marginal cost $\hat{c} > 1$ that corresponds to the point $\left(\hat{c}, \frac{2}{(1+\beta)\phi}\right)$ on the
stable path. Second, if the stable path reaches \( c = 1 \) at exactly \( k = 2(1+\beta )\varphi \), then the indifference point is \( c = 1 \). Third, for lower values of \( k \) on the stable path at \( c = 1 \), the value of the Hamiltonian evaluated along the stable path in the zero-production region is negative, such that the point of indifference (if at all) must be in the region with positive production.

As for the Full cartel the Hamilton function and the state-control system in the region with zero production are identical with their R&D cartel counterparts, this proof is valid also for the Full cartel regime.

\[ \mathbf{J}_{FC} = \begin{bmatrix} \frac{\partial \dot{c}}{\partial c} & \frac{\partial \dot{c}}{\partial k} \\ \frac{\partial \dot{k}}{\partial c} & \frac{\partial \dot{k}}{\partial k} \end{bmatrix} = \begin{bmatrix} 1 - (1+\beta )\varphi k & -(1+\beta )\varphi c \\ -(1+\beta )\phi (1-2c) \end{bmatrix}, \]  

At \( c = k = 0 \), the trace \( \tau \) of the matrix \( \mathbf{J}_{FC} \) is given as

\[ \tau \text{ def } \text{tr } \mathbf{J}_{FC} = 1 + \rho > 0, \]

its determinant \( \Delta \) is

\[ \Delta \text{ def } \text{det } \mathbf{J}_{FC} = \rho > 0, \]

and its discriminant \( D \) is

\[ D \text{ def } \tau^2 - 4\Delta = (1 - \rho)^2 > 0. \]

Hence, this steady state is an unstable node.

Evaluating the Jacobian matrix at

\[ k = \frac{(1+\beta )\phi }{8\rho } c(1-c), \quad c = \frac{1}{2} + V = \frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{32\rho }{(1+\beta )^2\varphi^2}}, \]

we obtain \( \tau = \rho > 0 \) and

\[ \Delta = \frac{\sqrt{(1+\beta )^2\varphi^2 - 32\rho } ((1+\beta )\phi + \sqrt{(1+\beta )^2\varphi^2 - 32\rho })}{16} \]

\[ \Delta = (1+\beta )^2\varphi^2 V \left( \frac{1}{8} + \frac{1}{4} V \right), \]

which is clearly positive if \( \phi > \sqrt{32\rho }/(1 + \beta ) \). Hence, this steady state is also unstable. The discriminant takes the value

\[ D = \rho (8 + \rho ) - \frac{1}{4} (1+\beta )\phi \left( (1+\beta )\phi + \sqrt{(1+\beta )^2\varphi^2 - 32\rho } \right), \]
which is zero for \( \phi = \phi_0 = \sqrt{\frac{2\rho(8+\rho)^2}{(1+\beta)^2(4+\rho)}} \), negative for \( \phi > \phi_0 \), and positive for \( \sqrt{32\rho/(1+\beta)} \leq \phi < \phi_0 \). The steady state is an unstable node if \( D > 0 \) and an unstable focus if \( D < 0 \). In the latter case, the eigenvalues of \( J^{FC} \) are complex conjugates with positive real parts.\(^{54}\)

Finally, evaluating the Jacobian matrix at

\[
k = \frac{(1 + \beta)\phi}{8 \rho} c(1 - c), \quad c = \frac{1}{2} - V = \frac{1}{2} - \frac{1}{2} \sqrt{1 - \frac{32\rho}{(1 + \beta)^2 \phi^2}},
\]

we obtain \( \tau = \rho > 0 \) and

\[
\Delta = \frac{(1 + \beta)^2 \phi^2 - 32\rho - (1 + \beta)\phi \sqrt{(1 + \beta)^2 \phi^2 - 32\rho}}{16}.
\]

If \( \phi > \sqrt{32\rho/(1+\beta)} \), then \( \Delta < 0 \), and the eigenvalues are real and have opposite sign. Therefore, (104) is a saddle-point steady state of the system. Observe that for \( \phi = \sqrt{32\rho/(1+\beta)} \), the steady states (100) and (104) coincide at \( c = 1/2 \) and \( k = \frac{1}{(1+\beta)\phi} \). A saddle-node bifurcation occurs at these parameter values, where the two equilibria coincide and disappear.

Substituting the expression for the steady-state marginal cost of the steady states other than the origin into (32), the expression for the optimal investment in the steady state simplifies to

\[
k^{FC} = \frac{1}{(1+\tau)\phi}.
\]

\(^{54}\)Note that the eigenvalues of \( J^{FC} \) are \( r_{1,2} = \frac{1}{2} (\tau \pm \sqrt{\tau^2 - 4\Delta}) \).